

Chen Shuwen's Notebooks (Part I)

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(1st Edition, 2021/8/16)

Chapter 1. The Generalization of Ramanujan nested radicals

$$2 \cos \frac{355\pi}{113} = -\sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - 2 \cos \frac{355\pi}{113}}}}}}}}}}}}}}}}}}}}}}}$$

Chapter 2. The Generalization of Euler's formula

$$\begin{aligned}
 & (\cos \frac{0\pi}{11})^k + (\cos \frac{2\pi}{11})^k + (\cos \frac{2\pi}{11})^k + (\cos \frac{4\pi}{11})^k + (\cos \frac{4\pi}{11})^k + (\cos \frac{6\pi}{11})^k + (\cos \frac{6\pi}{11})^k + (\cos \frac{8\pi}{11})^k + (\cos \frac{8\pi}{11})^k + (\cos \frac{10\pi}{11})^k + (\cos \frac{10\pi}{11})^k \\
 & = (\cos \frac{1\pi}{11})^k + (\cos \frac{1\pi}{11})^k + (\cos \frac{3\pi}{11})^k + (\cos \frac{3\pi}{11})^k + (\cos \frac{5\pi}{11})^k + (\cos \frac{5\pi}{11})^k + (\cos \frac{7\pi}{11})^k + (\cos \frac{7\pi}{11})^k + (\cos \frac{9\pi}{11})^k + (\cos \frac{9\pi}{11})^k + (\cos \frac{11\pi}{11})^k \\
 & \quad (k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10)
 \end{aligned}$$

Chapter 3. The Generalization of Newton Identities

$$\begin{aligned}
 P_k &= a_1^k + a_2^k + a_3^k + a_4^k + a_5^k \quad (k = 1, 3, 5, 7, 9) \\
 G_1 &= -P_1 \\
 G_3 &= (P_1^3 - P_3)/3 \\
 G_5 &= (P_1^5 - P_5)/5 \\
 G_7 &= (P_1^7 - P_7)/7 \\
 G_9 &= (P_1^9 - P_9)/9 - (G_3^3/3)
 \end{aligned}
 \Rightarrow G_1^2 + 2 \frac{\begin{vmatrix} G_5 & 0 & 1 \\ G_7 & G_3 & G_1^2 \\ G_9 & G_5 & G_1^4 + G_3 G_1 \end{vmatrix}}{\begin{vmatrix} G_3 & 0 & 1 \\ G_5 & G_3 & G_1^2 \\ G_7 & G_5 & G_1^4 + G_3 G_1 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2$$

Chapter 4. The Generalization of Ramanujan Identity

$$\begin{aligned}
 P_k &= a_1^k + a_2^k + a_3^k - b_1^k - b_2^k - b_3^k \quad (k = 1, 2, 3, 4, 5, 6) \\
 S_1 &= (P_1)/1 \\
 S_2 &= (P_2 + S_1 P_1)/2 \\
 S_3 &= (P_3 + S_1 P_2 + S_2 P_1)/3 \\
 S_4 &= (P_4 + S_1 P_3 + S_2 P_2 + S_3 P_1)/4 \\
 S_5 &= (P_5 + S_1 P_4 + S_2 P_3 + S_3 P_2 + S_4 P_1)/5 \\
 S_6 &= (P_6 + S_1 P_5 + S_2 P_4 + S_3 P_3 + S_4 P_2 + S_5 P_1)/6
 \end{aligned}
 \Rightarrow \frac{\begin{vmatrix} S_3 & S_2 & S_1 \\ S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 \end{vmatrix}}{\begin{vmatrix} S_3 & S_2 & S_1 \\ S_4 & S_3 & S_2 \\ S_5 & S_4 & S_3 \end{vmatrix}} = a_1 + a_2 + a_3$$

Chapter 5. The Generalization of the Prouhet-Tarry-Escott Problem

$$\begin{aligned}
 & 22 \times 22 \times 33 \times 33 \times 56 \times 56 \times 84 \times 84 = 21 \times 24 \times 28 \times 42 \times 44 \times 66 \times 77 \times 88 \\
 & 22^k + 22^k + 33^k + 33^k + 56^k + 56^k + 84^k + 84^k = 21^k + 24^k + 28^k + 42^k + 44^k + 66^k + 77^k + 88^k \\
 & \quad (k = -3, -2, -1, 1, 2, 3)
 \end{aligned}$$

Chapter 1

The Generalization of Ramanujan nested radicals

Theorem 1.1. (Chen Shuwen, 2021/7/25)

Let p_i be odd primes, k_i be positive integers. Then

(1) for any integer $h > 1$,

$$2^{(2 \prod_{i=1}^h \frac{p_i-1}{2} p_i^{k_i-1})} - 1 \equiv 0 \left(\text{mod} \prod_{i=1}^h p_i^{k_i} \right)$$

(2) for $p_1 \equiv \pm 1 \pmod{8}$,

$$2^{\left(\frac{p_1-1}{2} p_1^{k_1-1}\right)} - 1 \equiv 0 \pmod{p_1^{k_1}}$$

(3) for $p_1 \equiv \pm 3 \pmod{8}$,

$$2^{\left(\frac{p_1-1}{2} p_1^{k_1-1}\right)} + 1 \equiv 0 \pmod{p_1^{k_1}}$$

Example 1.1.

$$2^{2^{\left(\frac{7-1}{2} * 7^0\right) \left(\frac{1031-1}{2} * 1031^1\right) \left(\frac{19-1}{2} * 19^2\right)}} - 1 \equiv 0 \pmod{(7 * 19^3 * 1031^2)}$$

$$2^{2^{(18*41)}} - 1 \equiv 0 \pmod{(37 * 83)}$$

$$2^{8*17} - 1 \equiv 0 \pmod{(17^2)}$$

$$2^{3*7^3} - 1 \equiv 0 \pmod{(7^4)}$$

$$2^{9*19} + 1 \equiv 0 \pmod{(19^2)}$$

$$2^{26} + 1 \equiv 0 \pmod{(53)}$$

$$2^{26*53^3} + 1 \equiv 0 \pmod{(53^4)}$$

Theorem 1.2. (Chen Shuwen, 2021/7/22, proved by using Theorem 1.1)

For any rational number $\frac{n}{m}$, both $\sin\left(\frac{n\pi}{m}\right)$ and $\cos\left(\frac{n\pi}{m}\right)$ can be represented as cyclic infinite nested square

roots of 2, of which the cyclic period is less than $\frac{m-1}{2}$.

Example 1.2.

$$2 \sin\left(\frac{\pi}{18}\right) = \sqrt{2 - \sqrt{2 + \sqrt{2 + 2 \sin\left(\frac{\pi}{18}\right)}}} \quad (\text{Ramanujan})$$

$$2 \cos \left(\frac{\pi}{11} \right) = \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - 2 \cos \left(\frac{\pi}{11} \right)}}}}} \quad (\text{Sivakumar Krishnamoorthi})$$

Example 1.3. (Chen Shuwen, July 2021)

$$2 \cos 0 = \sqrt{2 + 2 \cos 0}$$

$$2 \cos \frac{1\pi}{3} = \sqrt{2 - 2 \cos \frac{1\pi}{3}}$$

$$2 \cos \frac{1\pi}{5} = \sqrt{2 + \sqrt{2 - 2 \cos \frac{1\pi}{5}}}$$

$$2 \cos \frac{\pi}{7} = \sqrt{2 + \sqrt{2 - \sqrt{2 - 2 \cos \frac{\pi}{7}}}}$$

$$2 \cos \frac{\pi}{9} = \sqrt{2 + \sqrt{2 + \sqrt{2 - 2 \cos \frac{\pi}{9}}}}$$

$$2 \cos \frac{\pi}{13} = \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 - 2 \cos \frac{\pi}{13}}}}}}}$$

$$2 \cos \frac{\pi}{15} = \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 - 2 \cos \frac{\pi}{15}}}}}$$

$$2 \cos \frac{\pi}{17} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - 2 \cos \frac{\pi}{17}}}}}$$

$$2 \cos \frac{\pi}{19} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - 2 \cos \frac{\pi}{19}}}}}}}}}$$

$$2 \cos \frac{\pi}{21} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 - 2 \cos \frac{\pi}{21}}}}}}}$$

$$2 \cos \frac{\pi}{23} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - 2 \cos \frac{\pi}{23}}}}}}}}}}}}}}$$

$$2 \cos \frac{\pi}{25} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 - 2 \cos \frac{\pi}{25}}}}}}}}}}}}$$

$$2 \cos \frac{\pi}{27} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - 2 \cos \frac{\pi}{27}}}}}}}}}}}}$$

$$2 \cos \frac{\pi}{29} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 - 2 \cos \frac{\pi}{29}}}}}}}}}}}}}}}}$$

$$2 \cos \frac{\pi}{31} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 - 2 \cos \frac{\pi}{31}}}}}}$$

$$2 \cos \frac{\pi}{33} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \cos \frac{\pi}{33}}}}}}$$

$$2 \cos \frac{\pi}{35} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 - 2 \cos \frac{\pi}{35}}}}}}}}}}}}}}$$

$$2 \cos \frac{\pi}{37} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 - 2 \cos \frac{\pi}{37}}}}}}}}}}}}}}}}}}}}$$

$$2\cos\frac{\pi}{39} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - 2\cos\frac{\pi}{39}}}}}}}}}}}}}}$$

$$2\cos\frac{\pi}{41} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 - 2\cos\frac{\pi}{41}}}}}}}}}}}}$$

$$2\cos\frac{\pi}{43} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - 2\cos\frac{\pi}{43}}}}}}}}$$

$$2\cos\frac{\pi}{45} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 - 2\cos\frac{\pi}{45}}}}}}}}}}}}}}$$

$$2\cos\frac{\pi}{47} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 - 2\cos\frac{\pi}{47}}}}}}}}}}}}}}}}}}}}}}$$

$$2\cos\frac{\pi}{49} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 - 2\cos\frac{\pi}{49}}}}}}}}}}}}}}}}}}}}}}$$

Example 1.4. (Chen Shuwen, July 2021)

$$2\cos\frac{\pi}{13} = \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 - 2\cos\frac{\pi}{13}}}}}}}}$$

$$2\cos\frac{2\pi}{13} = \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 + 2\cos\frac{2\pi}{13}}}}}}}}$$

$$2\sin\frac{\pi}{13} = \sqrt{2 - 2\cos\frac{2\pi}{13}}$$

$$2\cos\frac{\pi}{26} = \sqrt{2 + 2\cos\frac{\pi}{13}}$$

$$2\cos\frac{\pi}{52} = \sqrt{2 + \sqrt{2 + 2\cos\frac{\pi}{13}}}$$

$$2\cos\frac{\pi}{104} = \sqrt{2 + \sqrt{2 + \sqrt{2 + 2\cos\frac{\pi}{13}}}}$$

$$2\sin\frac{\pi}{104} = \sqrt{2 - \sqrt{2 + \sqrt{2 + 2\cos\frac{\pi}{13}}}}$$

$$2\sin\frac{\pi}{208} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + 2\cos\frac{\pi}{13}}}}}$$

$$2\sin\frac{27\pi}{208} = \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 + 2\cos\frac{\pi}{13}}}}}$$

$$2\sin\frac{77\pi}{208} = \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 + 2\cos\frac{\pi}{13}}}}}$$

$$2\sin\frac{79\pi}{208} = \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + 2\cos\frac{\pi}{13}}}}}$$

Example 1.5. (Chen Shuwen, July 2021)

$$2\cos\frac{\pi}{19} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - 2\cos\frac{\pi}{19}}}}}}}}}$$

$$2\cos\frac{2\pi}{19} = \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 + 2\cos\frac{2\pi}{19}}}}}}}}}$$

$$2\cos\frac{3\pi}{19} = \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - 2\cos\frac{3\pi}{19}}}}}}}}}}}$$

$$2 \cos \frac{4\pi}{19} = \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 + 2\cos \frac{4\pi}{19}}}}}}}}}}}}$$

$$2 \cos \frac{5\pi}{19} = \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - 2\cos \frac{5\pi}{19}}}}}}}}}}}}$$

$$2 \cos \frac{6\pi}{19} = \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + 2\cos \frac{6\pi}{19}}}}}}}}}}}}$$

$$2 \cos \frac{7\pi}{19} = \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 - 2\cos \frac{7\pi}{19}}}}}}}}}}}}$$

$$2 \cos \frac{8\pi}{19} = \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + 2\cos \frac{8\pi}{19}}}}}}}}}}}}$$

$$2 \cos \frac{9\pi}{19} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - 2\cos \frac{9\pi}{19}}}}}}}}}}}}$$

Example 1.6. (Chen Shuwen, August 2021)

$$2 \cos \frac{355\pi}{113} = - \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 - 2\cos \frac{355\pi}{113}}}}}}}}}}}}}}}}$$

$$2 \cos \frac{\pi}{65537} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - 2\cos \frac{\pi}{65537}}}}}}}}}}}}}}}}}}}}$$

Identities 1.2. (Chen Shuwen, 2021/8/5)

$$\sqrt[3]{1+0 \times 1} \sqrt[3]{2+1 \times 2} \sqrt[3]{3+2 \times 3} \sqrt[3]{4+3 \times 4} \sqrt[3]{5+4 \times 5} \sqrt[3]{6+5 \times 6} \sqrt[3]{7+6 \times 7} \sqrt[3]{8+7 \times 8} \sqrt[3]{9+\dots} = 1$$

$$\sqrt[3]{2+1 \times 2} \sqrt[3]{3+2 \times 3} \sqrt[3]{4+3 \times 4} \sqrt[3]{5+4 \times 5} \sqrt[3]{6+5 \times 6} \sqrt[3]{7+6 \times 7} \sqrt[3]{8+7 \times 8} \sqrt[3]{9+\dots} = 2$$

$$\sqrt[3]{3+2 \times 3} \sqrt[3]{4+3 \times 4} \sqrt[3]{5+4 \times 5} \sqrt[3]{6+5 \times 6} \sqrt[3]{7+6 \times 7} \sqrt[3]{8+7 \times 8} \sqrt[3]{9+\dots} = 3$$

$$\sqrt[3]{4+3 \times 4} \sqrt[3]{5+4 \times 5} \sqrt[3]{6+5 \times 6} \sqrt[3]{7+6 \times 7} \sqrt[3]{8+7 \times 8} \sqrt[3]{9+\dots} = 4$$

$$\sqrt[3]{5+4 \times 5} \sqrt[3]{6+5 \times 6} \sqrt[3]{7+6 \times 7} \sqrt[3]{8+7 \times 8} \sqrt[3]{9+\dots} = 5$$

.....

Reference 1.1.

Chapter 2

The Generalization of Euler's formula

Formula 2.1. (Chen Shuwen, 2021/7/28)

$$\sum_{k=1}^n e^{(-1)^{k+1} \frac{k i \pi}{n}} + 1 = 0$$

(n is odd integer and $n \geq 1$)

Example 2.1. (Chen Shuwen, 2021/7/28)

$$e^{\frac{1i\pi}{1}} + 1 = 0 \quad (\text{Euler})$$

$$e^{\frac{1i\pi}{3}} + e^{-\frac{2i\pi}{3}} + e^{\frac{3i\pi}{3}} + 1 = 0$$

$$e^{\frac{1i\pi}{5}} + e^{-\frac{2i\pi}{5}} + e^{\frac{3i\pi}{5}} + e^{-\frac{4i\pi}{5}} + e^{\frac{5i\pi}{5}} + 1 = 0$$

$$e^{\frac{1i\pi}{7}} + e^{-\frac{2i\pi}{7}} + e^{\frac{3i\pi}{7}} + e^{-\frac{4i\pi}{7}} + e^{\frac{5i\pi}{7}} + e^{-\frac{6i\pi}{7}} + e^{\frac{7i\pi}{7}} + 1 = 0$$

$$e^{\frac{1i\pi}{9}} + e^{-\frac{2i\pi}{9}} + e^{\frac{3i\pi}{9}} + e^{-\frac{4i\pi}{9}} + e^{\frac{5i\pi}{9}} + e^{-\frac{6i\pi}{9}} + e^{\frac{7i\pi}{9}} + e^{-\frac{8i\pi}{9}} + e^{\frac{9i\pi}{9}} + 1 = 0$$

$$e^{\frac{1i\pi}{11}} + e^{-\frac{2i\pi}{11}} + e^{\frac{3i\pi}{11}} + e^{-\frac{4i\pi}{11}} + e^{\frac{5i\pi}{11}} + e^{-\frac{6i\pi}{11}} + e^{\frac{7i\pi}{11}} + e^{-\frac{8i\pi}{11}} + e^{\frac{9i\pi}{11}} + e^{-\frac{10i\pi}{11}} + e^{\frac{11i\pi}{11}} + 1 = 0$$

Formula 2.1.1. (Simplification of Formula 2.1. Chen Shuwen, 2021/7/28)

$$\sum_{k=1}^{n-1} e^{(-1)^{k+1} \frac{k i \pi}{n}} = 0$$

(n is odd integer and $n \geq 3$)

Example 2.1.1

$$e^{\frac{1i\pi}{3}} + e^{-\frac{2i\pi}{3}} = 0$$

$$e^{\frac{1i\pi}{5}} + e^{-\frac{2i\pi}{5}} + e^{\frac{3i\pi}{5}} + e^{-\frac{4i\pi}{5}} = 0$$

$$e^{\frac{1i\pi}{7}} + e^{-\frac{2i\pi}{7}} + e^{\frac{3i\pi}{7}} + e^{-\frac{4i\pi}{7}} + e^{\frac{5i\pi}{7}} + e^{-\frac{6i\pi}{7}} = 0$$

$$e^{\frac{1i\pi}{9}} + e^{-\frac{2i\pi}{9}} + e^{\frac{3i\pi}{9}} + e^{-\frac{4i\pi}{9}} + e^{\frac{5i\pi}{9}} + e^{-\frac{6i\pi}{9}} + e^{\frac{7i\pi}{9}} + e^{-\frac{8i\pi}{9}} = 0$$

$$e^{\frac{1i\pi}{11}} + e^{-\frac{2i\pi}{11}} + e^{\frac{3i\pi}{11}} + e^{-\frac{4i\pi}{11}} + e^{\frac{5i\pi}{11}} + e^{-\frac{6i\pi}{11}} + e^{\frac{7i\pi}{11}} + e^{-\frac{8i\pi}{11}} + e^{\frac{9i\pi}{11}} + e^{-\frac{10i\pi}{11}} = 0$$

Formula 2.2. (Chen Shuwen, 2021/7/6)

$$\sum_{k=1}^n (-1)^{k+1} e^{\frac{ki\pi}{n}} = 0$$

(n is odd integer and $n \geq 3$)

Example 2.2. (Chen Shuwen, 2021/7/6)

$$e^{\frac{i\pi}{3}} - e^{\frac{2i\pi}{3}} + e^{\frac{3i\pi}{3}} = 0$$

$$e^{\frac{i\pi}{5}} - e^{\frac{2i\pi}{5}} + e^{\frac{3i\pi}{5}} - e^{\frac{4i\pi}{5}} + e^{\frac{5i\pi}{5}} = 0$$

$$e^{\frac{i\pi}{7}} - e^{\frac{2i\pi}{7}} + e^{\frac{3i\pi}{7}} - e^{\frac{4i\pi}{7}} + e^{\frac{5i\pi}{7}} - e^{\frac{6i\pi}{7}} + e^{\frac{7i\pi}{7}} = 0$$

$$e^{\frac{i\pi}{9}} - e^{\frac{2i\pi}{9}} + e^{\frac{3i\pi}{9}} - e^{\frac{4i\pi}{9}} + e^{\frac{5i\pi}{9}} - e^{\frac{6i\pi}{9}} + e^{\frac{7i\pi}{9}} - e^{\frac{8i\pi}{9}} + e^{\frac{9i\pi}{9}} = 0$$

$$e^{\frac{i\pi}{11}} - e^{\frac{2i\pi}{11}} + e^{\frac{3i\pi}{11}} - e^{\frac{4i\pi}{11}} + e^{\frac{5i\pi}{11}} - e^{\frac{6i\pi}{11}} + e^{\frac{7i\pi}{11}} - e^{\frac{8i\pi}{11}} + e^{\frac{9i\pi}{11}} - e^{\frac{10i\pi}{11}} + e^{\frac{11i\pi}{11}} = 0$$

Formula 2.2.1. (Simplification of Formula 2.2. Chen Shuwen, 2021/7/6)

$$\sum_{k=1}^{n-1} (-1)^{k+1} e^{\frac{ki\pi}{n}} = 1$$

(n is odd integer and $n \geq 3$)

Example 2.2.1

$$e^{\frac{i\pi}{3}} - e^{\frac{2i\pi}{3}} = 1$$

$$e^{\frac{i\pi}{5}} - e^{\frac{2i\pi}{5}} + e^{\frac{3i\pi}{5}} - e^{\frac{4i\pi}{5}} = 1$$

$$e^{\frac{i\pi}{7}} - e^{\frac{2i\pi}{7}} + e^{\frac{3i\pi}{7}} - e^{\frac{4i\pi}{7}} + e^{\frac{5i\pi}{7}} - e^{\frac{6i\pi}{7}} = 1$$

$$e^{\frac{i\pi}{9}} - e^{\frac{2i\pi}{9}} + e^{\frac{3i\pi}{9}} - e^{\frac{4i\pi}{9}} + e^{\frac{5i\pi}{9}} - e^{\frac{6i\pi}{9}} + e^{\frac{7i\pi}{9}} - e^{\frac{8i\pi}{9}} = 1$$

Formula 2.3. (Euler, Infinite analysis Introduction)

$$\cos z + \sum_{k=1}^{\frac{n}{2}-1} (-1)^k \left(\cos\left(\frac{k\pi}{n} + z\right) + \cos\left(\frac{k\pi}{n} - z\right) \right) = 0$$

(n is odd integer and $n \geq 3$, z is real number.)

It is equivalent to

$$\sum_{k=0}^{n-1} (-1)^k \cos\left(\frac{k\pi}{n} + z\right) = 0$$

Example 2.3. (Euler)

$$\cos z - \cos\left(\frac{\pi}{3} + z\right) + \cos\left(\frac{2\pi}{3} + z\right) = 0$$

$$\cos z - \cos\left(\frac{\pi}{5} + z\right) + \cos\left(\frac{2\pi}{5} + z\right) - \cos\left(\frac{3\pi}{5} + z\right) + \cos\left(\frac{4\pi}{5} + z\right) = 0$$

$$\cos z - \cos\left(\frac{\pi}{7} + z\right) + \cos\left(\frac{2\pi}{7} + z\right) - \cos\left(\frac{3\pi}{7} + z\right) + \cos\left(\frac{4\pi}{7} + z\right) - \cos\left(\frac{5\pi}{7} + z\right) + \cos\left(\frac{6\pi}{7} + z\right) = 0$$

Example 2.3.1 ($z = 0$)

$$\cos\left(\frac{1\pi}{3}\right) = \frac{1}{2}$$

$$\cos\left(\frac{1\pi}{5}\right) + \cos\left(\frac{3\pi}{5}\right) = \frac{1}{2}$$

$$\cos\left(\frac{1\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) + \cos\left(\frac{5\pi}{7}\right) = \frac{1}{2}$$

$$\cos\left(\frac{1\pi}{9}\right) + \cos\left(\frac{3\pi}{9}\right) + \cos\left(\frac{5\pi}{9}\right) + \cos\left(\frac{7\pi}{9}\right) = \frac{1}{2}$$

$$\cos\left(\frac{1\pi}{11}\right) + \cos\left(\frac{3\pi}{11}\right) + \cos\left(\frac{5\pi}{11}\right) + \cos\left(\frac{7\pi}{11}\right) + \cos\left(\frac{9\pi}{11}\right) = \frac{1}{2}$$

$$\cos\left(\frac{1\pi}{13}\right) + \cos\left(\frac{3\pi}{13}\right) + \cos\left(\frac{5\pi}{13}\right) + \cos\left(\frac{7\pi}{13}\right) + \cos\left(\frac{9\pi}{13}\right) + \cos\left(\frac{11\pi}{13}\right) = \frac{1}{2}$$

Formula 2.4. (Chen Shuwen, 2021/7/7)

$$\sum_{k=0}^{n-1} (-1)^k \left(\cos\left(\frac{k\pi}{n} + z\right)\right)^h = 0$$

(n is odd integer and $n \geq 3$, z is real number. $h = 1, 3, \dots, n - 2$)

Example 2.4.1 (Chen Shuwen, July 2021)

$$\cos^1\left(\frac{1\pi}{7} + z\right) + \cos^1\left(\frac{3\pi}{7} + z\right) + \cos^1\left(\frac{5\pi}{7} + z\right) = \cos^1\left(\frac{0\pi}{7} + z\right) + \cos^1\left(\frac{2\pi}{7} + z\right) + \cos^1\left(\frac{4\pi}{7} + z\right) + \cos^1\left(\frac{6\pi}{7} + z\right)$$

$$\cos^3\left(\frac{1\pi}{7} + z\right) + \cos^3\left(\frac{3\pi}{7} + z\right) + \cos^3\left(\frac{5\pi}{7} + z\right) = \cos^3\left(\frac{0\pi}{7} + z\right) + \cos^3\left(\frac{2\pi}{7} + z\right) + \cos^3\left(\frac{4\pi}{7} + z\right) + \cos^3\left(\frac{6\pi}{7} + z\right)$$

$$\cos^5\left(\frac{1\pi}{7} + z\right) + \cos^5\left(\frac{3\pi}{7} + z\right) + \cos^5\left(\frac{5\pi}{7} + z\right) = \cos^5\left(\frac{0\pi}{7} + z\right) + \cos^5\left(\frac{2\pi}{7} + z\right) + \cos^5\left(\frac{4\pi}{7} + z\right) + \cos^5\left(\frac{6\pi}{7} + z\right)$$

Example 2.4.2 (Chen Shuwen, July 2021)

$$\cos^1\left(\frac{1\pi}{7}\right) + \cos^1\left(\frac{3\pi}{7}\right) + \cos^1\left(\frac{5\pi}{7}\right) = \cos^1\left(\frac{0\pi}{7}\right) + \cos^1\left(\frac{2\pi}{7}\right) + \cos^1\left(\frac{4\pi}{7}\right) + \cos^1\left(\frac{6\pi}{7}\right) = \frac{1}{2}$$

$$\cos^3\left(\frac{1\pi}{7}\right) + \cos^3\left(\frac{3\pi}{7}\right) + \cos^3\left(\frac{5\pi}{7}\right) = \cos^3\left(\frac{0\pi}{7}\right) + \cos^3\left(\frac{2\pi}{7}\right) + \cos^3\left(\frac{4\pi}{7}\right) + \cos^3\left(\frac{6\pi}{7}\right) = \frac{1}{2}$$

$$\cos^5\left(\frac{1\pi}{7}\right) + \cos^5\left(\frac{3\pi}{7}\right) + \cos^5\left(\frac{5\pi}{7}\right) = \cos^5\left(\frac{0\pi}{7}\right) + \cos^5\left(\frac{2\pi}{7}\right) + \cos^5\left(\frac{4\pi}{7}\right) + \cos^5\left(\frac{6\pi}{7}\right) = \frac{1}{2}$$

Example 2.4.3 (Chen Shuwen, July 2021)

$$\begin{aligned} \cos\left(\frac{\pi}{27}\right)^1 + \cos\left(\frac{5\pi}{27}\right)^1 + \cos\left(\frac{7\pi}{27}\right)^1 + \cos\left(\frac{11\pi}{27}\right)^1 + \cos\left(\frac{13\pi}{27}\right)^1 &= \cos\left(\frac{2\pi}{27}\right)^1 + \cos\left(\frac{4\pi}{27}\right)^1 + \cos\left(\frac{8\pi}{27}\right)^1 + \cos\left(\frac{10\pi}{27}\right)^1 \\ \cos\left(\frac{\pi}{27}\right)^3 + \cos\left(\frac{5\pi}{27}\right)^3 + \cos\left(\frac{7\pi}{27}\right)^3 + \cos\left(\frac{11\pi}{27}\right)^3 + \cos\left(\frac{13\pi}{27}\right)^3 &= \cos\left(\frac{2\pi}{27}\right)^3 + \cos\left(\frac{4\pi}{27}\right)^3 + \cos\left(\frac{8\pi}{27}\right)^3 + \cos\left(\frac{10\pi}{27}\right)^3 \\ \cos\left(\frac{\pi}{27}\right)^5 + \cos\left(\frac{5\pi}{27}\right)^5 + \cos\left(\frac{7\pi}{27}\right)^5 + \cos\left(\frac{11\pi}{27}\right)^5 + \cos\left(\frac{13\pi}{27}\right)^5 &= \cos\left(\frac{2\pi}{27}\right)^5 + \cos\left(\frac{4\pi}{27}\right)^5 + \cos\left(\frac{8\pi}{27}\right)^5 + \cos\left(\frac{10\pi}{27}\right)^5 \\ \cos\left(\frac{\pi}{27}\right)^7 + \cos\left(\frac{5\pi}{27}\right)^7 + \cos\left(\frac{7\pi}{27}\right)^7 + \cos\left(\frac{11\pi}{27}\right)^7 + \cos\left(\frac{13\pi}{27}\right)^7 &= \cos\left(\frac{2\pi}{27}\right)^7 + \cos\left(\frac{4\pi}{27}\right)^7 + \cos\left(\frac{8\pi}{27}\right)^7 + \cos\left(\frac{10\pi}{27}\right)^7 \end{aligned}$$

Example 2.4.4 (Chen Shuwen, July 2021)

$$\begin{aligned} \sin^k\left(\frac{\pi}{54}\right) + \sin^k\left(\frac{5\pi}{54}\right) + \sin^k\left(\frac{7\pi}{54}\right) + \sin^k\left(\frac{11\pi}{54}\right) + \sin^k\left(\frac{13\pi}{54}\right) + \sin^k\left(\frac{17\pi}{54}\right) + \sin^k\left(\frac{19\pi}{54}\right) + \sin^k\left(\frac{23\pi}{54}\right) + \sin^k\left(\frac{25\pi}{54}\right) \\ = \sin^k\left(\frac{2\pi}{54}\right) + \sin^k\left(\frac{4\pi}{54}\right) + \sin^k\left(\frac{8\pi}{54}\right) + \sin^k\left(\frac{10\pi}{54}\right) + \sin^k\left(\frac{14\pi}{54}\right) + \sin^k\left(\frac{16\pi}{54}\right) + \sin^k\left(\frac{20\pi}{54}\right) + \sin^k\left(\frac{22\pi}{54}\right) + \sin^k\left(\frac{26\pi}{54}\right) \\ (k = 2, 4, 6, 8, 10, 12, 14, 16) \end{aligned}$$

Example 2.4.5 (Chen Shuwen, 2021/7/4)

$$\begin{aligned} (\cos\frac{0\pi}{11})^k + (\cos\frac{2\pi}{11})^k + (\cos\frac{2\pi}{11})^k + (\cos\frac{4\pi}{11})^k + (\cos\frac{4\pi}{11})^k + (\cos\frac{6\pi}{11})^k + (\cos\frac{6\pi}{11})^k + (\cos\frac{8\pi}{11})^k + (\cos\frac{8\pi}{11})^k + (\cos\frac{10\pi}{11})^k + (\cos\frac{10\pi}{11})^k \\ = (\cos\frac{1\pi}{11})^k + (\cos\frac{1\pi}{11})^k + (\cos\frac{3\pi}{11})^k + (\cos\frac{3\pi}{11})^k + (\cos\frac{5\pi}{11})^k + (\cos\frac{5\pi}{11})^k + (\cos\frac{7\pi}{11})^k + (\cos\frac{7\pi}{11})^k + (\cos\frac{9\pi}{11})^k + (\cos\frac{9\pi}{11})^k + (\cos\frac{11\pi}{11})^k \\ (k=1,2,3,4,5,6,7,8,9,10) \end{aligned}$$

Example 2.4.6 (Chen Shuwen, 2021/7/4)

$$\begin{aligned} \{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}\} \\ = \{\cos^2\left(\frac{1\pi}{22}\right), \cos^2\left(\frac{1\pi}{22}\right), \cos^2\left(\frac{3\pi}{22}\right), \cos^2\left(\frac{3\pi}{22}\right), \cos^2\left(\frac{5\pi}{22}\right), \cos^2\left(\frac{5\pi}{22}\right), \cos^2\left(\frac{7\pi}{22}\right), \cos^2\left(\frac{7\pi}{22}\right), \cos^2\left(\frac{9\pi}{22}\right), \cos^2\left(\frac{9\pi}{22}\right), \cos^2\left(\frac{11\pi}{22}\right)\} \\ \{b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}\} \\ = \{\cos^2\left(\frac{0\pi}{22}\right), \cos^2\left(\frac{2\pi}{22}\right), \cos^2\left(\frac{2\pi}{22}\right), \cos^2\left(\frac{4\pi}{22}\right), \cos^2\left(\frac{4\pi}{22}\right), \cos^2\left(\frac{6\pi}{22}\right), \cos^2\left(\frac{6\pi}{22}\right), \cos^2\left(\frac{8\pi}{22}\right), \cos^2\left(\frac{8\pi}{22}\right), \cos^2\left(\frac{10\pi}{22}\right), \cos^2\left(\frac{10\pi}{22}\right)\} \\ a_0^k + a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k + a_7^k + a_8^k + a_9^k + a_{10}^k = b_0^k + b_1^k + b_2^k + b_3^k + b_4^k + b_5^k + b_6^k + b_7^k + b_8^k + b_9^k + b_{10}^k \\ (k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10) \end{aligned}$$

Example 2.4.7 (Chen Shuwen, 2021/7/7)

$$\begin{aligned} \{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}\} \\ = \{\sin^2\left(\frac{\pi}{88}\right), \sin^2\left(\frac{7\pi}{88}\right), \sin^2\left(\frac{9\pi}{88}\right), \sin^2\left(\frac{15\pi}{88}\right), \sin^2\left(\frac{17\pi}{88}\right), \sin^2\left(\frac{23\pi}{88}\right), \sin^2\left(\frac{25\pi}{88}\right), \sin^2\left(\frac{31\pi}{88}\right), \sin^2\left(\frac{33\pi}{88}\right), \sin^2\left(\frac{39\pi}{88}\right), \sin^2\left(\frac{41\pi}{88}\right)\} \\ \{b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}\} \\ = \{\sin^2\left(\frac{3\pi}{88}\right), \sin^2\left(\frac{5\pi}{88}\right), \sin^2\left(\frac{11\pi}{88}\right), \sin^2\left(\frac{13\pi}{88}\right), \sin^2\left(\frac{19\pi}{88}\right), \sin^2\left(\frac{21\pi}{88}\right), \sin^2\left(\frac{27\pi}{88}\right), \sin^2\left(\frac{29\pi}{88}\right), \sin^2\left(\frac{35\pi}{88}\right), \sin^2\left(\frac{37\pi}{88}\right), \sin^2\left(\frac{43\pi}{88}\right)\} \\ a_0^k + a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k + a_7^k + a_8^k + a_9^k + a_{10}^k \\ = b_0^k + b_1^k + b_2^k + b_3^k + b_4^k + b_5^k + b_6^k + b_7^k + b_8^k + b_9^k + b_{10}^k \\ (k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10) \end{aligned}$$

Example 2.4.8 (Chen Shuwen, July 2021)

$$\left(\sin \frac{5\pi}{28}\right)^1 + \left(\sin \frac{7\pi}{28}\right)^1 + \left(\sin \frac{13\pi}{28}\right)^1 = \left(\sin \frac{\pi}{28}\right)^1 + \left(\sin \frac{3\pi}{28}\right)^1 + \left(\sin \frac{9\pi}{28}\right)^1 + \left(\sin \frac{11\pi}{28}\right)^1$$

$$\left(\sin \frac{5\pi}{28}\right)^3 + \left(\sin \frac{7\pi}{28}\right)^3 + \left(\sin \frac{13\pi}{28}\right)^3 = \left(\sin \frac{\pi}{28}\right)^3 + \left(\sin \frac{3\pi}{28}\right)^3 + \left(\sin \frac{9\pi}{28}\right)^3 + \left(\sin \frac{11\pi}{28}\right)^3$$

$$\left(\sin \frac{5\pi}{28}\right)^5 + \left(\sin \frac{7\pi}{28}\right)^5 + \left(\sin \frac{13\pi}{28}\right)^5 = \left(\sin \frac{\pi}{28}\right)^5 + \left(\sin \frac{3\pi}{28}\right)^5 + \left(\sin \frac{9\pi}{28}\right)^5 + \left(\sin \frac{11\pi}{28}\right)^5$$

$$\left(\sin \frac{\pi}{32}\right)^2 + \left(\sin \frac{7\pi}{32}\right)^2 + \left(\sin \frac{9\pi}{32}\right)^2 + \left(\sin \frac{15\pi}{32}\right)^2 = \left(\sin \frac{3\pi}{32}\right)^2 + \left(\sin \frac{5\pi}{32}\right)^2 + \left(\sin \frac{11\pi}{32}\right)^2 + \left(\sin \frac{13\pi}{32}\right)^2$$

$$\left(\sin \frac{\pi}{32}\right)^4 + \left(\sin \frac{7\pi}{32}\right)^4 + \left(\sin \frac{9\pi}{32}\right)^4 + \left(\sin \frac{15\pi}{32}\right)^4 = \left(\sin \frac{3\pi}{32}\right)^4 + \left(\sin \frac{5\pi}{32}\right)^4 + \left(\sin \frac{11\pi}{32}\right)^4 + \left(\sin \frac{13\pi}{32}\right)^4$$

$$\left(\sin \frac{\pi}{32}\right)^6 + \left(\sin \frac{7\pi}{32}\right)^6 + \left(\sin \frac{9\pi}{32}\right)^6 + \left(\sin \frac{15\pi}{32}\right)^6 = \left(\sin \frac{3\pi}{32}\right)^6 + \left(\sin \frac{5\pi}{32}\right)^6 + \left(\sin \frac{11\pi}{32}\right)^6 + \left(\sin \frac{13\pi}{32}\right)^6$$

$$\left(\sin \frac{5\pi}{36}\right)^1 + \left(\sin \frac{7\pi}{36}\right)^1 + \left(\sin \frac{13\pi}{36}\right)^1 + \left(\sin \frac{15\pi}{36}\right)^1 = \left(\sin \frac{\pi}{36}\right)^1 + \left(\sin \frac{3\pi}{36}\right)^1 + \left(\sin \frac{9\pi}{36}\right)^1 + \left(\sin \frac{11\pi}{36}\right)^1 + \left(\sin \frac{17\pi}{36}\right)^1$$

$$\left(\sin \frac{5\pi}{36}\right)^3 + \left(\sin \frac{7\pi}{36}\right)^3 + \left(\sin \frac{13\pi}{36}\right)^3 + \left(\sin \frac{15\pi}{36}\right)^3 = \left(\sin \frac{\pi}{36}\right)^3 + \left(\sin \frac{3\pi}{36}\right)^3 + \left(\sin \frac{9\pi}{36}\right)^3 + \left(\sin \frac{11\pi}{36}\right)^3 + \left(\sin \frac{17\pi}{36}\right)^3$$

$$\left(\sin \frac{5\pi}{36}\right)^5 + \left(\sin \frac{7\pi}{36}\right)^5 + \left(\sin \frac{13\pi}{36}\right)^5 + \left(\sin \frac{15\pi}{36}\right)^5 = \left(\sin \frac{\pi}{36}\right)^5 + \left(\sin \frac{3\pi}{36}\right)^5 + \left(\sin \frac{9\pi}{36}\right)^5 + \left(\sin \frac{11\pi}{36}\right)^5 + \left(\sin \frac{17\pi}{36}\right)^5$$

$$\left(\sin \frac{5\pi}{36}\right)^7 + \left(\sin \frac{7\pi}{36}\right)^7 + \left(\sin \frac{13\pi}{36}\right)^7 + \left(\sin \frac{15\pi}{36}\right)^7 = \left(\sin \frac{\pi}{36}\right)^7 + \left(\sin \frac{3\pi}{36}\right)^7 + \left(\sin \frac{9\pi}{36}\right)^7 + \left(\sin \frac{11\pi}{36}\right)^7 + \left(\sin \frac{17\pi}{36}\right)^7$$

Example 2.4.9 (Chen Shuwen, August 2021)

$$\{ a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{17}, a_{18} \} = \{ \sin^2 \left(\frac{\pi}{1729} \right), \sin^2 \left(\frac{90\pi}{1729} \right), \sin^2 \left(\frac{92\pi}{1729} \right),$$

$$\sin^2 \left(\frac{181\pi}{1729} \right), \sin^2 \left(\frac{183\pi}{1729} \right), \sin^2 \left(\frac{272\pi}{1729} \right), \sin^2 \left(\frac{274\pi}{1729} \right), \sin^2 \left(\frac{363\pi}{1729} \right), \sin^2 \left(\frac{365\pi}{1729} \right), \sin^2 \left(\frac{454\pi}{1729} \right), \sin^2 \left(\frac{456\pi}{1729} \right),$$

$$\sin^2 \left(\frac{545\pi}{1729} \right), \sin^2 \left(\frac{547\pi}{1729} \right), \sin^2 \left(\frac{636\pi}{1729} \right), \sin^2 \left(\frac{638\pi}{1729} \right), \sin^2 \left(\frac{727\pi}{1729} \right), \sin^2 \left(\frac{729\pi}{1729} \right), \sin^2 \left(\frac{818\pi}{1729} \right), \sin^2 \left(\frac{820\pi}{1729} \right) \}$$

$$\{ b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, b_{13}, b_{14}, b_{15}, b_{16}, b_{17}, b_{18} \} = \{ \cos^2 \left(\frac{\pi}{1729} \right), \cos^2 \left(\frac{90\pi}{1729} \right), \cos^2 \left(\frac{92\pi}{1729} \right),$$

$$\cos^2 \left(\frac{181\pi}{1729} \right), \cos^2 \left(\frac{183\pi}{1729} \right), \cos^2 \left(\frac{272\pi}{1729} \right), \cos^2 \left(\frac{274\pi}{1729} \right), \cos^2 \left(\frac{363\pi}{1729} \right), \cos^2 \left(\frac{365\pi}{1729} \right), \cos^2 \left(\frac{454\pi}{1729} \right), \cos^2 \left(\frac{456\pi}{1729} \right),$$

$$\cos^2 \left(\frac{545\pi}{1729} \right), \cos^2 \left(\frac{547\pi}{1729} \right), \cos^2 \left(\frac{636\pi}{1729} \right), \cos^2 \left(\frac{638\pi}{1729} \right), \cos^2 \left(\frac{727\pi}{1729} \right), \cos^2 \left(\frac{729\pi}{1729} \right), \cos^2 \left(\frac{818\pi}{1729} \right), \cos^2 \left(\frac{820\pi}{1729} \right) \}$$

$$a_0^k + a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k + a_7^k + a_8^k + a_9^k + a_{10}^k + a_{11}^k + a_{12}^k + a_{13}^k + a_{14}^k + a_{15}^k + a_{16}^k + a_{17}^k + a_{18}^k = b_0^k + b_1^k + b_2^k + b_3^k + b_4^k + b_5^k + b_6^k + b_7^k + b_8^k + b_9^k + b_{10}^k + b_{11}^k + b_{12}^k + b_{13}^k + b_{14}^k + b_{15}^k + b_{16}^k + b_{17}^k + b_{18}^k$$

$$(k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18)$$

Formula 2.5.1 (Chen Shuwen, 2021/8/12)For positive integer n

$$\frac{1}{2} + \sum_{j=1}^{n-1} \sin^{2k} \left(\frac{\pi j}{2n} \right) = \frac{1}{2} + \sum_{j=1}^{n-1} \cos^{2k} \left(\frac{\pi j}{2n} \right) = \frac{n(2k-1)!}{2^{2k-1} k! (k-1)!} = \frac{n\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k+1)}$$

$$(k = 1, 2, 3, \dots, 2n-1)$$

Formula 2.5.2 (Chen Shuwen, 2021/8/12)For positive odd integer n and any z

$$\sin^{2k}(z) + \sum_{j=1}^{\frac{n-1}{2}} \left(\sin^{2k} \left(\frac{\pi j}{n} - z \right) + \sin^{2k} \left(\frac{\pi j}{n} + z \right) \right) = 2 \sum_{j=1}^{\frac{n-1}{2}} \sin^{2k} \left(\frac{\pi j}{n} \right) = \frac{n(2k-1)!}{2^{2k-1} k! (k-1)!} = \frac{n\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k+1)}$$

$$\cos^{2k}(z) + \sum_{j=1}^{\frac{n-1}{2}} \left(\cos^{2k} \left(\frac{\pi j}{n} - z \right) + \cos^{2k} \left(\frac{\pi j}{n} + z \right) \right) = 1 + 2 \sum_{j=1}^{\frac{n-1}{2}} \cos^{2k} \left(\frac{\pi j}{n} \right) = \frac{n(2k-1)!}{2^{2k-1} k! (k-1)!} = \frac{n\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k+1)}$$

$$(k = 1, 2, 3, \dots, n-1)$$

Formula 2.5.3 (Chen Shuwen, 2021/8/12)For positive odd integer n and any z

$$\cos^k(z) + \sum_{j=1}^{\frac{n-1}{2}} \left(\cos^k \left(\frac{2\pi j}{n} - z \right) + \cos^k \left(\frac{2\pi j}{n} + z \right) \right) = 1 + 2 \sum_{j=1}^{\frac{n-1}{2}} \cos^k \left(\frac{2\pi j}{n} \right)$$

$$(k = 1, 2, 3, 4, \dots, n-2, n-1, n+1, n+3, n+5, \dots, 2n-2)$$

It is equivalent to

$$\cos^k(z) + \sum_{j=1}^{\frac{n-1}{2}} \left(\cos^k \left(\frac{2\pi j}{n} - z \right) + \cos^k \left(\frac{2\pi j}{n} + z \right) \right) = 1 + 2 \sum_{j=1}^{\frac{n-1}{2}} \cos^k \left(\frac{2\pi j}{n} \right) = 0 \quad (k = 1, 3, 5, \dots, n-2)$$

$$\cos^k(z) + \sum_{j=1}^{\frac{n-1}{2}} \left(\cos^k \left(\frac{2\pi j}{n} - z \right) + \cos^k \left(\frac{2\pi j}{n} + z \right) \right) = 1 + 2 \sum_{j=1}^{\frac{n-1}{2}} \cos^k \left(\frac{2\pi j}{n} \right) = \frac{n(k-1)!}{2^{k-1} (\frac{k}{2}-1)! \frac{k}{2}!} = \frac{n\Gamma(\frac{k+1}{2})}{\sqrt{\pi}\Gamma(\frac{k}{2}+1)}$$

$$(k = 2, 4, 6, \dots, 2n-2)$$

Formula 2.5.4 (Chen Shuwen, 2021/8/12)For positive even integer n and any z

$$\sum_{j=1}^{\frac{n}{2}} \left(\sin^{2k} \left(\frac{\pi(2j-1)}{2n} - z \right) + \sin^{2k} \left(\frac{\pi(2j-1)}{2n} + z \right) \right) = 2 \sum_{j=1}^{\frac{n}{2}} \sin^{2k} \left(\frac{\pi(2j-1)}{2n} \right) = \frac{n(2k-1)!}{2^{2k-1} k! (k-1)!} = \frac{n\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k+1)}$$

Example 2.5.4 (Chen Shuwen, August 2021)

$$\begin{aligned}
 & \cos^k\left(\frac{0\pi}{55}\right) + \cos^k\left(\frac{10\pi}{55}\right) + \cos^k\left(\frac{10\pi}{55}\right) + \cos^k\left(\frac{20\pi}{55}\right) + \cos^k\left(\frac{20\pi}{55}\right) + \cos^k\left(\frac{30\pi}{55}\right) + \cos^k\left(\frac{30\pi}{55}\right) + \cos^k\left(\frac{40\pi}{55}\right) + \cos^k\left(\frac{40\pi}{55}\right) + \cos^k\left(\frac{50\pi}{55}\right) + \cos^k\left(\frac{50\pi}{55}\right) \\
 &= \cos^k\left(\frac{1\pi}{55}\right) + \cos^k\left(\frac{9\pi}{55}\right) + \cos^k\left(\frac{11\pi}{55}\right) + \cos^k\left(\frac{19\pi}{55}\right) + \cos^k\left(\frac{21\pi}{55}\right) + \cos^k\left(\frac{29\pi}{55}\right) + \cos^k\left(\frac{31\pi}{55}\right) + \cos^k\left(\frac{39\pi}{55}\right) + \cos^k\left(\frac{41\pi}{55}\right) + \cos^k\left(\frac{49\pi}{55}\right) + \cos^k\left(\frac{51\pi}{55}\right) \\
 &= \cos^k\left(\frac{2\pi}{55}\right) + \cos^k\left(\frac{8\pi}{55}\right) + \cos^k\left(\frac{12\pi}{55}\right) + \cos^k\left(\frac{18\pi}{55}\right) + \cos^k\left(\frac{22\pi}{55}\right) + \cos^k\left(\frac{28\pi}{55}\right) + \cos^k\left(\frac{32\pi}{55}\right) + \cos^k\left(\frac{38\pi}{55}\right) + \cos^k\left(\frac{42\pi}{55}\right) + \cos^k\left(\frac{48\pi}{55}\right) + \cos^k\left(\frac{52\pi}{55}\right) \\
 &= \cos^k\left(\frac{3\pi}{55}\right) + \cos^k\left(\frac{7\pi}{55}\right) + \cos^k\left(\frac{13\pi}{55}\right) + \cos^k\left(\frac{17\pi}{55}\right) + \cos^k\left(\frac{23\pi}{55}\right) + \cos^k\left(\frac{27\pi}{55}\right) + \cos^k\left(\frac{33\pi}{55}\right) + \cos^k\left(\frac{37\pi}{55}\right) + \cos^k\left(\frac{43\pi}{55}\right) + \cos^k\left(\frac{47\pi}{55}\right) + \cos^k\left(\frac{53\pi}{55}\right) \\
 &= \cos^k\left(\frac{4\pi}{55}\right) + \cos^k\left(\frac{6\pi}{55}\right) + \cos^k\left(\frac{14\pi}{55}\right) + \cos^k\left(\frac{16\pi}{55}\right) + \cos^k\left(\frac{24\pi}{55}\right) + \cos^k\left(\frac{26\pi}{55}\right) + \cos^k\left(\frac{34\pi}{55}\right) + \cos^k\left(\frac{36\pi}{55}\right) + \cos^k\left(\frac{44\pi}{55}\right) + \cos^k\left(\frac{46\pi}{55}\right) + \cos^k\left(\frac{54\pi}{55}\right) \\
 &= \cos^k\left(\frac{5\pi}{55}\right) + \cos^k\left(\frac{5\pi}{55}\right) + \cos^k\left(\frac{15\pi}{55}\right) + \cos^k\left(\frac{15\pi}{55}\right) + \cos^k\left(\frac{25\pi}{55}\right) + \cos^k\left(\frac{25\pi}{55}\right) + \cos^k\left(\frac{35\pi}{55}\right) + \cos^k\left(\frac{35\pi}{55}\right) + \cos^k\left(\frac{45\pi}{55}\right) + \cos^k\left(\frac{45\pi}{55}\right) + \cos^k\left(\frac{55\pi}{55}\right) \\
 &= \cos^k\left(\frac{1\pi}{22}\right) + \cos^k\left(\frac{3\pi}{22}\right) + \cos^k\left(\frac{5\pi}{22}\right) + \cos^k\left(\frac{7\pi}{22}\right) + \cos^k\left(\frac{9\pi}{22}\right) + \cos^k\left(\frac{11\pi}{22}\right) + \cos^k\left(\frac{13\pi}{22}\right) + \cos^k\left(\frac{15\pi}{22}\right) + \cos^k\left(\frac{17\pi}{22}\right) + \cos^k\left(\frac{19\pi}{22}\right) + \cos^k\left(\frac{21\pi}{22}\right) \\
 &= \cos^k\left(\frac{1\pi}{33}\right) + \cos^k\left(\frac{5\pi}{33}\right) + \cos^k\left(\frac{7\pi}{33}\right) + \cos^k\left(\frac{11\pi}{33}\right) + \cos^k\left(\frac{13\pi}{33}\right) + \cos^k\left(\frac{17\pi}{33}\right) + \cos^k\left(\frac{19\pi}{33}\right) + \cos^k\left(\frac{23\pi}{33}\right) + \cos^k\left(\frac{25\pi}{33}\right) + \cos^k\left(\frac{29\pi}{33}\right) + \cos^k\left(\frac{31\pi}{33}\right) \\
 &= \cos^k\left(\frac{2\pi}{33}\right) + \cos^k\left(\frac{4\pi}{33}\right) + \cos^k\left(\frac{8\pi}{33}\right) + \cos^k\left(\frac{10\pi}{33}\right) + \cos^k\left(\frac{14\pi}{33}\right) + \cos^k\left(\frac{16\pi}{33}\right) + \cos^k\left(\frac{20\pi}{33}\right) + \cos^k\left(\frac{22\pi}{33}\right) + \cos^k\left(\frac{26\pi}{33}\right) + \cos^k\left(\frac{28\pi}{33}\right) + \cos^k\left(\frac{32\pi}{33}\right) \\
 &= \cos^k\left(\frac{1\pi}{44}\right) + \cos^k\left(\frac{7\pi}{44}\right) + \cos^k\left(\frac{9\pi}{44}\right) + \cos^k\left(\frac{15\pi}{44}\right) + \cos^k\left(\frac{17\pi}{44}\right) + \cos^k\left(\frac{23\pi}{44}\right) + \cos^k\left(\frac{25\pi}{44}\right) + \cos^k\left(\frac{31\pi}{44}\right) + \cos^k\left(\frac{33\pi}{44}\right) + \cos^k\left(\frac{39\pi}{44}\right) + \cos^k\left(\frac{41\pi}{44}\right) \\
 &= \cos^k\left(\frac{3\pi}{44}\right) + \cos^k\left(\frac{5\pi}{44}\right) + \cos^k\left(\frac{11\pi}{44}\right) + \cos^k\left(\frac{13\pi}{44}\right) + \cos^k\left(\frac{19\pi}{44}\right) + \cos^k\left(\frac{21\pi}{44}\right) + \cos^k\left(\frac{27\pi}{44}\right) + \cos^k\left(\frac{29\pi}{44}\right) + \cos^k\left(\frac{35\pi}{44}\right) + \cos^k\left(\frac{37\pi}{44}\right) + \cos^k\left(\frac{43\pi}{44}\right) \\
 & \quad (k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 16, 18, 20)
 \end{aligned}$$

Example 2.5.5 (Chen Shuwen, August 2021)

$$\begin{aligned}
 a_1 &= \sqrt{10 + 2\sqrt{5}} & b_1 &= \sqrt{8 + 2\sqrt{10 + 2\sqrt{5}}} & c_1 &= \sqrt{8 - 2\sqrt{8 + 2\sqrt{10 + 2\sqrt{5}}}} & d_1 &= -\sqrt{8 - 2\sqrt{8 - 2\sqrt{8 + 2\sqrt{10 + 2\sqrt{5}}}}} \\
 a_2 &= -\sqrt{10 - 2\sqrt{5}} & b_2 &= \sqrt{8 - 2\sqrt{10 - 2\sqrt{5}}} & c_2 &= \sqrt{8 - 2\sqrt{8 - 2\sqrt{10 - 2\sqrt{5}}}} & d_2 &= \sqrt{8 - 2\sqrt{8 - 2\sqrt{8 - 2\sqrt{10 - 2\sqrt{5}}}}} \\
 a_3 &= 0 & b_3 &= -\sqrt{8} & c_3 &= -\sqrt{8 + 4\sqrt{2}} & d_3 &= \sqrt{8 + 2\sqrt{8 + 4\sqrt{2}}} \\
 a_4 &= \sqrt{10 - 2\sqrt{5}} & b_4 &= -\sqrt{8 + 2\sqrt{10 - 2\sqrt{5}}} & c_4 &= \sqrt{8 + 2\sqrt{8 + 2\sqrt{10 - 2\sqrt{5}}}} & d_4 &= \sqrt{8 - 2\sqrt{8 + 2\sqrt{8 + 2\sqrt{10 - 2\sqrt{5}}}}} \\
 a_5 &= -\sqrt{10 + 2\sqrt{5}} & b_5 &= \sqrt{8 - 2\sqrt{10 + 2\sqrt{5}}} & c_5 &= -\sqrt{8 - 2\sqrt{8 - 2\sqrt{10 + 2\sqrt{5}}}} & d_5 &= -\sqrt{8 + 2\sqrt{8 - 2\sqrt{8 - 2\sqrt{10 + 2\sqrt{5}}}}} \\
 a_1^k + a_2^k + a_3^k + a_4^k + a_5^k &= b_1^k + b_2^k + b_3^k + b_4^k + b_5^k = c_1^k + c_2^k + c_3^k + c_4^k + c_5^k = d_1^k + d_2^k + d_3^k + d_4^k + d_5^k \\
 & \quad (k = 1, 2, 3, 4, 6, 8)
 \end{aligned}$$

Chapter 3

The Generalization of Newton Identities

Identities 3.1. Equivalent form of Newton's binomial theorem (Chen Shuwen, 1985)

$$(a + b)^n = \sum_{j=0}^n h_j a^j b^{n-j}$$

here

$$h_0 = 1$$

$$h_j = h_{j-1} \frac{(n+1-j)}{j} \quad (j \geq 1)$$

Example 3.1. Simplification of Pascal's Triangle (Chen Shuwen, 1985, by using Identities 3.1)

For $(a + b)^9$

Step 1

$$[\] \frac{9}{1} [\] \frac{8}{2} [\] \frac{7}{3} [\] \frac{6}{4} [\] \frac{5}{5} [\] - [\] - [\] - [\] - [\]$$

Step 2

$$[1] \frac{9}{1} [9] \frac{8}{2} [36] \frac{7}{3} [84] \frac{6}{4} [126] \frac{5}{5} [\] - [\] - [\] - [\] - [\]$$

$$(Note: 1 \times \frac{9}{1} = 9, \quad 9 \times \frac{8}{2} = 36, \quad 36 \times \frac{7}{3} = 84, \quad 84 \times \frac{6}{4} = 126 \)$$

Step 3

$$[1] \frac{9}{1} [9] \frac{8}{2} [36] \frac{7}{3} [84] \frac{6}{4} [126] \frac{5}{5} [126] - [84] - [36] - [9] - [1]$$

Then we have

$$(a + b)^9 = a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 + 126a^4b^5 + 84a^3b^6 + 36a^2b^7 + 9ab^8 + b^9$$

Identities 3.2. A Generalization of Newton's binomial theorem (Chen Shuwen, 1985)

$$(a + b + c)^n = \sum_{i=0}^n \sum_{j=0}^i h_{i,j} a^j b^{i-j} c^{n-i}$$

here

$$h_{0,0} = 1$$

$$h_{i,0} = \frac{(n+1-i)h_{i-1,0}}{i} \quad (1 \leq i \leq n)$$

$$h_{i,j} = \frac{(i+1-j)h_{i,j-1}}{j} \quad (1 \leq j \leq i)$$

Remark. Identities 3.1 and Identities 3.2 can be generalized to $(a_1 + a_2 + \dots + a_m)^n$ for any m .

Example 3.2. The Generalization of Pascal's Triangle (Chen Shuwen, 1985, by using Identities 3.2)

For $(a + b + c)^8$

Step 1. Fill in the bottom row for $(a + b + 0)^8$, same as $(a + b)^8$:

$$[1] \frac{8}{1} [8] \frac{7}{2} [28] \frac{6}{3} [56] \frac{5}{4} [70] \frac{4}{5} [56] \frac{3}{6} [28] \frac{2}{7} [8] \frac{1}{8} [1]$$

Step 2. Fill in the first number of each row, same data as the bottom row:

$$\begin{aligned}
 & [1] \\
 & [8] - [\quad] \\
 & [28] - [\quad] - [\quad] \\
 & [56] - [\quad] - [\quad] - [\quad] \\
 & [70] - [\quad] - [\quad] - [\quad] - [\quad] \\
 & [56] - [\quad] - [\quad] - [\quad] - [\quad] - [\quad] \\
 & [28] - [\quad] - [\quad] - [\quad] - [\quad] - [\quad] - [\quad] \\
 & [8] - [\quad] - [\quad] - [\quad] - [\quad] - [\quad] - [\quad] - [\quad] \\
 & [1] \frac{8}{1} [8] \frac{7}{2} [28] \frac{6}{3} [56] \frac{5}{4} [70] \frac{4}{5} [56] \frac{3}{6} [28] \frac{2}{7} [8] \frac{1}{8} [1]
 \end{aligned}$$

Step 3. Fill in the rest numbers, same method as the bottom row:

$$\begin{aligned}
 & [1] \\
 & [8] \frac{1}{1} [8] \\
 & [28] \frac{2}{1} [56] \frac{1}{2} [28] \\
 & [56] \frac{3}{1} [168] \frac{2}{2} [168] \frac{1}{3} [56] \\
 & [70] \frac{4}{1} [280] \frac{3}{2} [420] \frac{2}{3} [280] \frac{1}{4} [70] \\
 & [56] \frac{5}{1} [280] \frac{4}{2} [560] \frac{3}{3} [560] \frac{2}{4} [280] \frac{1}{5} [56] \\
 & [28] \frac{6}{1} [168] \frac{5}{2} [420] \frac{4}{3} [560] \frac{3}{4} [420] \frac{2}{5} [168] \frac{1}{6} [28] \\
 & [8] \frac{7}{1} [56] \frac{6}{2} [168] \frac{5}{3} [280] \frac{4}{4} [280] \frac{3}{5} [168] \frac{2}{6} [56] \frac{1}{7} [8] \\
 & [1] \frac{8}{1} [8] \frac{7}{2} [28] \frac{6}{3} [56] \frac{5}{4} [70] \frac{4}{5} [56] \frac{3}{6} [28] \frac{2}{7} [8] \frac{1}{8} [1]
 \end{aligned}$$

Then we have

$$\begin{aligned}
 (a + b + c)^8 &= c^8 \\
 &+ 8ac^7 + 8bc^7
 \end{aligned}$$

$$\begin{aligned}
 &+28a^2c^6 + 56abc^6 + 28b^2c^6 \\
 &+56a^3c^5 + 168a^2bc^5 + 168ab^2c^5 + 56b^3c^5 \\
 &+70a^4c^4 + 280a^3bc^4 + 420a^2b^2c^4 + 280ab^3c^4 + 70b^4c^4 \\
 &+56a^5c^3 + 280a^4bc^3 + 560a^3b^2c^3 + 560a^2b^3c^3 + 280ab^4c^3 + 56b^5c^3 \\
 &+28a^6c^2 + 168a^5bc^2 + 420a^4b^2c^2 + 560a^3b^3c^2 + 420a^2b^4c^2 + 168ab^5c^2 + 28b^6c^2 \\
 &+8a^7c + 56a^6bc + 168a^5b^2c + 280a^4b^3c + 280a^3b^4c + 168a^2b^5c + 56ab^6c + 8b^7c \\
 &+a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8
 \end{aligned}$$

Identities 3.3. (The Girard-Newton Identities, Classical form, by Girard and Newton)

Let n and k be positive integer. Denote

$$P_k = a_1^k + a_2^k + a_3^k + \cdots + a_n^k$$

and

$$\begin{aligned}
 S_1 &= -(a_1 + a_2 + a_3 + \cdots + a_n) \\
 S_2 &= a_1a_2 + a_1a_3 + a_2a_3 + \cdots + a_{n-1}a_n \\
 S_3 &= -(a_1a_2a_3 + \cdots + a_{n-2}a_{n-1}a_n) \\
 &\vdots \\
 S_n &= (-1)^n(a_1a_2a_3 \dots a_n)
 \end{aligned}$$

then

$$\begin{aligned}
 P_1 + 1S_1 &= 0 \\
 P_2 + S_1P_1 + 2S_2 &= 0 \\
 P_3 + S_1P_2 + S_2P_1 + 3S_3 &= 0 \\
 &\vdots \\
 P_n + S_1P_{n-1} + S_2P_{n-2} + \cdots + S_{n-1}P_1 + nS_n &= 0 \\
 P_k + S_1P_{k-1} + S_2P_{k-2} + \cdots + S_{n-1}P_{k-n+1} + S_nP_{k-n} &= 0 \quad \text{for } k > n
 \end{aligned}$$

Identities 3.4. (Equivalent form of The Girard-Newton Identities, by Chen Shuwen, 2017, 2019)

Let n and k be positive integer. Denote

$$P_k = a_1^k + a_2^k + a_3^k + \cdots + a_n^k$$

and denote

$$\begin{aligned}
 S_1 &= -(P_1)/1 \\
 S_2 &= -(P_2 + S_1P_1)/2 \\
 S_3 &= -(P_3 + S_1P_2 + S_2P_1)/3 \\
 &\vdots \\
 S_k &= -(P_k + S_1P_{k-1} + S_2P_{k-2} + \cdots + S_{k-1}P_1)/k
 \end{aligned}$$

Then

$$\begin{aligned}
 S_1 &= -(a_1 + a_2 + a_3 + \cdots + a_n) \\
 S_2 &= a_1a_2 + a_1a_3 + a_2a_3 + \cdots + a_{n-1}a_n \\
 S_3 &= -(a_1a_2a_3 + \cdots + a_{n-2}a_{n-1}a_n) \\
 &\vdots \\
 S_k &= (-1)^k \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} a_{j_1}a_{j_2} \dots a_{j_k} \\
 &\vdots \\
 S_n &= (-1)^n(a_1a_2a_3 \dots a_n) \\
 S_k &= 0 \quad \text{for } k > n
 \end{aligned}$$

and

$$\begin{aligned}
 P_1 + 1S_1 &= 0 \\
 P_2 + S_1P_1 + 2S_2 &= 0 \\
 P_3 + S_1P_2 + S_2P_1 + 3S_3 &= 0 \\
 &\vdots \\
 P_n + S_1P_{n-1} + S_2P_{n-2} + \cdots + S_{n-1}P_1 + nS_n &= 0 \\
 P_k + S_1P_{k-1} + S_2P_{k-2} + \cdots + S_{n-1}P_{k-n+1} + S_nP_{k-n} &= 0 \quad \text{for } k > n
 \end{aligned}$$

Example 3.3. ($n = 5$ case of Identities 3.4)

Let k be positive integer. Denote

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k$$

and Denote

$$\begin{aligned}
 S_1 &= -(P_1)/1 \\
 S_2 &= -(P_2 + S_1P_1)/2 \\
 S_3 &= -(P_3 + S_1P_2 + S_2P_1)/3 \\
 S_4 &= -(P_4 + S_1P_3 + S_2P_2 + S_3P_1)/4 \\
 S_5 &= -(P_5 + S_1P_4 + S_2P_3 + S_3P_2 + S_4P_1)/5 \\
 S_6 &= -(P_6 + S_1P_5 + S_2P_4 + S_3P_3 + S_4P_2 + S_5P_1)/6 \\
 S_7 &= -(P_7 + S_1P_6 + S_2P_5 + S_3P_4 + S_4P_3 + S_5P_2 + S_6P_1)/7 \\
 S_8 &= -(P_8 + S_1P_7 + S_2P_6 + S_3P_5 + S_4P_4 + S_5P_3 + S_6P_2 + S_7P_1)/8 \\
 &\vdots \\
 S_k &= -(P_k + S_1P_{k-1} + S_2P_{k-2} + \cdots + S_{k-1}P_1)/k
 \end{aligned}$$

Then

$$\begin{aligned}
 S_1 &= -(a_1 + a_2 + a_3 + a_4 + a_5) \\
 S_2 &= a_1a_2 + a_1a_3 + a_2a_3 + \cdots + a_4a_5 \\
 S_3 &= -(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + \cdots + a_3a_4a_5) \\
 S_4 &= a_1a_2a_3a_4 + a_1a_2a_3a_5 + a_1a_2a_4a_5 + a_1a_3a_4a_5 + a_2a_3a_4a_5 \\
 S_5 &= -a_1a_2a_3a_4a_5 \\
 S_6 &= 0 \\
 S_7 &= 0 \\
 S_8 &= 0 \\
 &\vdots \\
 S_k &= 0 \quad \text{for } k > 5
 \end{aligned}$$

and

$$\begin{aligned}
 P_1 + 1S_1 &= 0 \\
 P_2 + S_1P_1 + 2S_2 &= 0 \\
 P_3 + S_1P_2 + S_2P_1 + 3S_3 &= 0 \\
 P_4 + S_1P_3 + S_2P_2 + S_3P_1 + 4S_4 &= 0 \\
 P_5 + S_1P_4 + S_2P_3 + S_3P_2 + S_4P_1 + 5S_5 &= 0 \\
 P_6 + S_1P_5 + S_2P_4 + S_3P_3 + S_4P_2 + S_5P_1 &= 0 \\
 P_7 + S_1P_6 + S_2P_5 + S_3P_4 + S_4P_3 + S_5P_2 &= 0 \\
 P_8 + S_1P_7 + S_2P_6 + S_3P_5 + S_4P_4 + S_5P_3 &= 0 \\
 &\vdots \\
 P_k + S_1P_{k-1} + S_2P_{k-2} + S_3P_{k-3} + S_4P_{k-4} + S_5P_{k-5} &= 0 \quad \text{for } k > 5
 \end{aligned}$$

Identities 3.5. (Chen Shuwen, 2017, 2019)

Let

$$S_1 = -|P_1| / 1!$$

$$S_2 = \begin{vmatrix} P_1 & 1 \\ P_2 & P_1 \end{vmatrix} / 2!$$

$$S_3 = - \begin{vmatrix} P_1 & 1 & 0 \\ P_2 & P_1 & 2 \\ P_3 & P_2 & P_1 \end{vmatrix} / 3!$$

$$S_4 = \begin{vmatrix} P_1 & 1 & 0 & 0 \\ P_2 & P_1 & 2 & 0 \\ P_3 & P_2 & P_1 & 3 \\ P_4 & P_3 & P_2 & P_1 \end{vmatrix} / 4!$$

$$S_5 = - \begin{vmatrix} P_1 & 1 & 0 & 0 & 0 \\ P_2 & P_1 & 2 & 0 & 0 \\ P_3 & P_2 & P_1 & 3 & 0 \\ P_4 & P_3 & P_2 & P_1 & 4 \\ P_5 & P_4 & P_3 & P_2 & P_1 \end{vmatrix} / 5!$$

$$S_6 = \begin{vmatrix} P_1 & 1 & 0 & 0 & 0 & 0 \\ P_2 & P_1 & 2 & 0 & 0 & 0 \\ P_3 & P_2 & P_1 & 3 & 0 & 0 \\ P_4 & P_3 & P_2 & P_1 & 4 & 0 \\ P_5 & P_4 & P_3 & P_2 & P_1 & 5 \\ P_6 & P_5 & P_4 & P_3 & P_2 & P_1 \end{vmatrix} / 6!$$

$$S_7 = - \begin{vmatrix} P_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ P_2 & P_1 & 2 & 0 & 0 & 0 & 0 \\ P_3 & P_2 & P_1 & 3 & 0 & 0 & 0 \\ P_4 & P_3 & P_2 & P_1 & 4 & 0 & 0 \\ P_5 & P_4 & P_3 & P_2 & P_1 & 5 & 0 \\ P_6 & P_5 & P_4 & P_3 & P_2 & P_1 & 6 \\ P_7 & P_6 & P_5 & P_4 & P_3 & P_2 & P_1 \end{vmatrix} / 7!$$

$$S_8 = \begin{vmatrix} P_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_2 & P_1 & 2 & 0 & 0 & 0 & 0 & 0 \\ P_3 & P_2 & P_1 & 3 & 0 & 0 & 0 & 0 \\ P_4 & P_3 & P_2 & P_1 & 4 & 0 & 0 & 0 \\ P_5 & P_4 & P_3 & P_2 & P_1 & 5 & 0 & 0 \\ P_6 & P_5 & P_4 & P_3 & P_2 & P_1 & 6 & 0 \\ P_7 & P_6 & P_5 & P_4 & P_3 & P_2 & P_1 & 7 \\ P_8 & P_7 & P_6 & P_5 & P_4 & P_3 & P_2 & P_1 \end{vmatrix} / 8!$$

$$S_k = (-1)^k \begin{vmatrix} P_1 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ P_2 & P_1 & 2 & \dots & 0 & 0 & 0 & 0 \\ P_3 & P_2 & P_1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ P_{k-2} & P_{k-3} & P_{k-4} & \dots & P_1 & k-2 & 0 & 0 \\ P_{k-1} & P_{k-2} & P_{k-3} & \dots & P_2 & P_1 & k-1 & 0 \\ P_k & P_{k-1} & P_{k-2} & \dots & P_3 & P_2 & P_1 & 0 \end{vmatrix} / k!$$

then the above relations are equivalent to

$$S_1 = -(P_1)/1$$

$$S_2 = -(P_2 + S_1P_1)/2$$

$$S_3 = -(P_3 + S_1P_2 + S_2P_1)/3$$

$$S_4 = -(P_4 + S_1P_3 + S_2P_2 + S_3P_1)/4$$

$$S_5 = -(P_5 + S_1P_4 + S_2P_3 + S_3P_2 + S_4P_1)/5$$

$$S_6 = -(P_6 + S_1P_5 + S_2P_4 + S_3P_3 + S_4P_2 + S_5P_1)/6$$

$$S_7 = -(P_7 + S_1P_6 + S_2P_5 + S_3P_4 + S_4P_3 + S_5P_2 + S_6P_1)/7$$

$$S_8 = -(P_8 + S_1P_7 + S_2P_6 + S_3P_5 + S_4P_4 + S_5P_3 + S_6P_2 + S_7P_1)/8$$

⋮

$$S_k = -(P_k + S_1P_{k-1} + S_2P_{k-2} + \dots + S_{k-1}P_1)/k$$

Definition 3.1. (P form of S_k)

By expand the below relations of S_k and P_k

$$S_1 = -(P_1)/1$$

$$S_2 = -(P_2 + S_1P_1)/2$$

$$S_3 = -(P_3 + S_1P_2 + S_2P_1)/3$$

⋮

$$S_k = -(P_k + S_1P_{k-1} + S_2P_{k-2} + \dots + S_{k-1}P_1)/k$$

We have

$$\begin{aligned}
 S_1 &= -P_1 \\
 S_2 &= \frac{P_1^2}{2} - \frac{P_2}{2} \\
 S_3 &= -\frac{P_1^3}{6} + \frac{P_1P_2}{2} - \frac{P_3}{3} \\
 S_4 &= \frac{P_1^4}{24} - \frac{1}{4}P_1^2P_2 + \frac{P_2^2}{8} + \frac{P_1P_3}{3} - \frac{P_4}{4} \\
 S_5 &= -\frac{P_1^5}{120} + \frac{1}{12}P_1^3P_2 - \frac{1}{8}P_1P_2^2 - \frac{1}{6}P_1^2P_3 + \frac{P_2P_3}{6} + \frac{P_1P_4}{4} - \frac{P_5}{5} \\
 S_6 &= \frac{P_1^6}{720} - \frac{1}{48}P_1^4P_2 + \frac{1}{16}P_1^2P_2^2 - \frac{P_2^3}{48} + \frac{1}{18}P_1^3P_3 - \frac{1}{6}P_1P_2P_3 + \frac{P_3^2}{18} - \frac{1}{8}P_1^2P_4 + \frac{P_2P_4}{8} + \frac{P_1P_5}{5} - \frac{P_6}{6} \\
 S_7 &= -\frac{P_1^7}{5040} + \frac{1}{240}P_1^5P_2 - \frac{1}{48}P_1^3P_2^2 + \frac{1}{48}P_1P_2^3 - \frac{1}{72}P_1^4P_3 + \frac{1}{12}P_1^2P_2P_3 - \frac{1}{24}P_2^2P_3 - \frac{1}{18}P_1P_3^2 + \frac{1}{24}P_1^3P_4 \\
 &\quad - \frac{1}{8}P_1P_2P_4 + \frac{P_3P_4}{12} - \frac{1}{10}P_1^2P_5 + \frac{P_2P_5}{10} + \frac{P_1P_6}{6} - \frac{P_7}{7} \\
 &\dots\dots
 \end{aligned}$$

Generally

$$S_k = \sum_{\substack{0 \leq i_1, i_2, i_3, \dots, i_k \leq k \\ i_1 + 2i_2 + \dots + ki_k = k}} \frac{(-P_1/1)^{i_1} (-P_2/2)^{i_2} \dots (-P_k/k)^{i_k}}{i_1! i_2! \dots i_k!}$$

We call it the P form of S_k .

Definition 3.2. (Q form of S_k , Chen Shuwen, 2017)

For the relation of S_k and P_k

$$\begin{aligned}
 S_1 &= -(P_1)/1 \\
 S_2 &= -(P_2 + S_1P_1)/2 \\
 S_3 &= -(P_3 + S_1P_2 + S_2P_1)/3 \\
 &\vdots \\
 S_k &= -(P_k + S_1P_{k-1} + S_2P_{k-2} + \dots + S_{k-1}P_1)/k
 \end{aligned}$$

If we let

$$P_k = -k Q_k$$

then

$$\begin{aligned}
 S_1 &= Q_1 \\
 S_2 &= \frac{Q_1^2}{2!} + Q_2 \\
 S_3 &= \frac{Q_1^3}{3!} + Q_1Q_2 + Q_3 \\
 S_4 &= \frac{Q_1^4}{4!} + \frac{Q_1^2}{2!}Q_2 + \frac{Q_2^2}{2!} + Q_1Q_3 + Q_4 \\
 S_5 &= \frac{Q_1^5}{5!} + \frac{Q_1^3}{3!}Q_2 + \frac{Q_2^2}{2!}Q_1 + \frac{Q_1^2}{2!}Q_3 + Q_2Q_3 + Q_1Q_4 + Q_5 \\
 S_6 &= \frac{Q_1^6}{6!} + \frac{Q_1^4}{4!}Q_2 + \frac{Q_2^2}{2!}Q_1^2 + \frac{Q_2^3}{3!} + \frac{Q_1^3}{3!}Q_3 + Q_1Q_2Q_3 + \frac{Q_2^3}{2!} + \frac{Q_1^2}{2!}Q_4 + Q_2Q_4 + Q_1Q_5 + Q_6
 \end{aligned}$$

$$S_7 = \frac{Q_1^7}{7!} + \frac{Q_1^5}{5!} Q_2 + \frac{Q_1^3}{3!} \frac{Q_2^2}{2!} + Q_1 \frac{Q_2^3}{3!} + \frac{Q_1^4}{4!} Q_3 + \frac{Q_1^2}{2!} Q_2 Q_3 + \frac{Q_2^2}{2!} Q_3 + Q_1 \frac{Q_3^2}{2!} + \frac{Q_1^3}{3!} Q_4 + Q_1 Q_2 Q_4 + Q_3 Q_4$$

$$+ \frac{Q_1^2}{2!} Q_5 + Q_2 Q_5 + Q_1 Q_6 + Q_7$$

... ..

Generally

$$S_k = \sum_{\substack{0 \leq i_1, i_2, i_3, \dots, i_k \leq k \\ i_1 + 2i_2 + \dots + ki_k = k}} \frac{Q_1^{i_1} Q_2^{i_2} \dots Q_k^{i_k}}{i_1! i_2! \dots i_k!}$$

We call it the Q form of S_k . In case of Q form, the sign of each item is always positive.

Definition 3.3. (F form of S_k , Chen Shuwen, 1999)

For the relation of S_k and P_k

$$S_1 = -(P_1)/1$$

$$S_2 = -(P_2 + S_1 P_1)/2$$

$$S_3 = -(P_3 + S_1 P_2 + S_2 P_1)/3$$

$$\vdots$$

$$S_k = -(P_k + S_1 P_{k-1} + S_2 P_{k-2} + \dots + S_{k-1} P_1)/k$$

If we let

$$P_1 = -F_1$$

$$P_k = P_1^k - kF_k \quad \text{for } k > 1$$

then

$$S_1 = F_1$$

$$S_2 = F_2$$

$$S_3 = F_1 F_2 + F_3$$

$$S_4 = \frac{F_2^2}{2} + F_1 F_3 + F_4$$

$$S_5 = \frac{1}{2} F_1 F_2^2 + F_2 F_3 + F_1 F_4 + F_5$$

$$S_6 = \frac{F_2^3}{6} + F_1 F_2 F_3 + \frac{F_3^2}{2} + F_2 F_4 + F_1 F_5 + F_6$$

$$S_7 = \frac{1}{6} F_1 F_2^3 + \frac{1}{2} F_2^2 F_3 + \frac{1}{2} F_1 F_3^2 + F_1 F_2 F_4 + F_3 F_4 + F_2 F_5 + F_1 F_6 + F_7$$

$$S_8 = \frac{F_2^4}{24} + \frac{1}{2} F_1 F_2^2 F_3 + \frac{1}{2} F_2 F_3^2 + \frac{1}{2} F_2^2 F_4 + F_1 F_3 F_4 + \frac{F_4^2}{2} + F_1 F_2 F_5 + F_3 F_5 + F_2 F_6 + F_1 F_7 + F_8$$

... ..

Generally

$$S_k = \sum_{\substack{0 \leq i_1 \leq 1 \\ 0 \leq i_2, i_3, \dots, i_k \leq k \\ i_1 + 2i_2 + \dots + ki_k = k}} \frac{F_1^{i_1} F_2^{i_2} F_3^{i_3} \dots F_k^{i_k}}{i_2! i_3! \dots i_k!}$$

We call it the F form of S_k . In case of F form, the sign of each item is positive, and the power exponent of F_1 is always 1.

Definition 3.4. (*D* form of S_k , Chen Shuwen, 2020)

For the relation of S_k and P_k

$$\begin{aligned} S_1 &= -(P_1)/1 \\ S_2 &= -(P_2 + S_1P_1)/2 \\ S_3 &= -(P_3 + S_1P_2 + S_2P_1)/3 \\ &\vdots \\ S_k &= -(P_k + S_1P_{k-1} + S_2P_{k-2} + \dots + S_{k-1}P_1)/k \end{aligned}$$

If we let

$$\begin{aligned} P_1 &= -D_1 \\ P_2 &= D_1^2 - 2D_2 \\ P_3 &= -D_1^3 - 3D_3 \\ P_4 &= D_1^4 + 2D_2^2 - 4D_4 \square \\ P_5 &= -D_1^5 - 5D_5 \\ P_6 &= D_1^6 - 2D_2^3 + 3D_3^2 - 6D_6 \\ P_7 &= -D_1^7 - 7D_7 \\ P_8 &= D_1^8 + 2D_2^4 + 4D_4^2 - 8D_8 \\ P_9 &= -D_1^9 - 3D_3^3 - 9D_9 \\ P_{10} &= D_1^{10} - 2D_2^5 + 5D_5^2 - 10D_{10} \\ P_{11} &= -D_1^{11} - 11D_{11} \\ &\vdots \end{aligned}$$

$$P_k = \sum_{j=1}^h (-1)^{\binom{k}{f_j}} f_j D_{f_j}^{\binom{k}{f_j}}$$

here f_1, f_2, \dots, f_h are all positive factors of k , include 1 and k itself.

then

$$\begin{aligned} S_1 &= D_1 \\ S_2 &= D_2 \\ S_3 &= D_1D_2 + D_3 \\ S_4 &= D_1D_3 + D_4 \\ S_5 &= D_2D_3 + D_1D_4 + D_5 \\ S_6 &= D_1D_2D_3 + D_2D_4 + D_1D_5 + D_6 \\ S_7 &= D_1D_2D_4 + D_3D_4 + D_2D_5 + D_1D_6 + D_7 \\ S_8 &= D_1D_3D_4 + D_1D_2D_5 + D_3D_5 + D_2D_6 + D_1D_7 + D_8 \\ S_9 &= D_2D_3D_4 + D_1D_3D_5 + D_4D_5 + D_1D_2D_6 + D_3D_6 + D_2D_7 + D_1D_8 + D_9 \\ S_{10} &= D_1D_2D_3D_4 + D_2D_3D_5 + D_1D_4D_5 + D_1D_3D_6 + D_4D_6 + D_1D_2D_7 + D_3D_7 + D_2D_8 + D_1D_9 + D_{10} \\ S_{11} &= D_1D_2D_3D_5 + D_2D_4D_5 + D_2D_3D_6 + D_1D_4D_6 + D_5D_6 + D_1D_3D_7 + D_4D_7 + D_1D_2D_8 + D_3D_8 + D_2D_9 \\ &\quad + D_1D_{10} + D_{11} \\ &\dots \end{aligned}$$

Generally

$$S_k = \sum_{\substack{0 \leq i_1, i_2, i_3, \dots, i_k \leq 1 \\ i_1 + 2i_2 + \dots + ki_k = k}} D_1^{i_1} D_2^{i_2} D_3^{i_3} \dots D_k^{i_k}$$

We call it the *D* form of S_k . In case of *D* form, the sign of each item is positive, the coefficient of each item is always 1, and the power exponents of D_1, D_2, \dots, D_k all are 1.

Example 3.3.

Comparison of P form, Q form, F form, and D form:

$$S_7 = -\frac{P_1^7}{5040} + \frac{1}{240}P_1^5P_2 - \frac{1}{48}P_1^3P_2^2 + \frac{1}{48}P_1P_2^3 - \frac{1}{72}P_1^4P_3 + \frac{1}{12}P_1^2P_2P_3 - \frac{1}{24}P_2^2P_3 - \frac{1}{18}P_1P_3^2 + \frac{1}{24}P_1^3P_4 - \frac{1}{8}P_1P_2P_4 + \frac{P_3P_4}{12} - \frac{1}{10}P_1^2P_5 + \frac{P_2P_5}{10} + \frac{P_1P_6}{6} - \frac{P_7}{7}$$

$$S_7 = \frac{Q_1^7}{7!} + \frac{Q_1^5}{5!}Q_2 + \frac{Q_1^3}{3!}\frac{Q_2^2}{2!} + Q_1\frac{Q_2^3}{3!} + \frac{Q_1^4}{4!}Q_3 + \frac{Q_1^2}{2!}Q_2Q_3 + \frac{Q_2^2}{2!}Q_3 + Q_1\frac{Q_3^2}{2!} + \frac{Q_1^3}{3!}Q_4 + Q_1Q_2Q_4 + Q_3Q_4 + \frac{Q_1^2}{2!}Q_5 + Q_2Q_5 + Q_1Q_6 + Q_7$$

$$S_7 = \frac{F_2^3}{3!}F_1 + \frac{F_2^2}{2!}F_3 + \frac{F_2^2}{2!}F_1 + F_1F_2F_4 + F_3F_4 + F_2F_5 + F_1F_6 + F_7$$

$$S_7 = D_1D_2D_4 + D_3D_4 + D_2D_5 + D_1D_6 + D_7$$

Definition 3.4.1. An equivalent definition of D form

The below relation in Definition 3.4.

$$P_k = \sum_{j=1}^h (-1)^{\binom{k}{f_j}} f_j D_{f_j}^{\binom{k}{f_j}} \quad \text{here } f_1, f_2, \dots, f_h \text{ are all positive factors of } k, \text{ include 1 and } k \text{ itself.}$$

is equivalent to

$$P_1 = -D_1$$

$$P_k = (-1)^k D_1^k - kD_k$$

k is prime number

$$P_k = (-1)^k D_1^k - kD_k + \sum_{j=1}^h (-1)^{\binom{k}{f_j}} f_j D_{f_j}^{\binom{k}{f_j}}$$

k is composite number with all its positive factors f_1, f_2, \dots, f_h , exclude 1 and k itself.

Example:

$$P_1 = -D_1$$

$$P_2 = D_1^2 - 2D_2$$

$$P_3 = -D_1^3 - 3D_3$$

$$P_4 = D_1^4 - 4D_4 + (2D_2^2) \square$$

$$P_5 = -D_1^5 - 5D_5$$

$$P_6 = D_1^6 - 6D_6 + (-2D_2^3 + 3D_3^2)$$

$$P_7 = -D_1^7 - 7D_7$$

$$P_8 = D_1^8 - 8D_8 + (2D_2^4 + 4D_4^2)$$

$$P_9 = -D_1^9 - 9D_9 + (-3D_3^3)$$

$$P_{10} = D_1^{10} - 10D_{10} + (-2D_2^5 + 5D_5^2)$$

$$P_{11} = -D_1^{11} - 11D_{11}$$

$$P_{12} = D_1^{12} - 12D_{12} + (2D_2^6 + 3D_3^4 - 4D_4^3 + 6D_6^2)$$

$$P_{13} = -D_1^{13} - 13D_{13}$$

$$P_{14} = D_1^{14} - 14D_{14} + (-2D_2^7 + 7D_7^2)$$

$$P_{15} = -D_1^{15} - 15D_{15} + (-3D_3^5 - 5D_5^3)$$

$$P_{16} = D_1^{16} - 16D_{16} + (2D_2^8 + 4D_4^4 + 8D_8^2)$$

$$P_{17} = -D_1^{17} - 17D_{17}$$

... ..

It is very interesting that the relation of P_k and D_k shows the primeness and factors of k .

Identities 3.6. (*P* form of The Girard-Newton Identities, by Chen Shuwen, 2017-2019)

Let n and k be positive integer. Denote

$$P_k = a_1^k + a_2^k + a_3^k + \cdots + a_n^k$$

and

$$S_k = \sum_{\substack{0 \leq i_1, i_2, i_3, \dots, i_k \leq k \\ i_1 + 2i_2 + \cdots + ki_k = k}} \frac{(-P_1/1)^{i_1} (-P_2/2)^{i_2} \cdots (-P_k/k)^{i_k}}{i_1! i_2! \cdots i_k!}$$

then

$$S_k = (-1)^k \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} a_{j_1} a_{j_2} \cdots a_{j_k} \quad k \leq n$$

$$S_k = 0 \quad k > n$$

Identities 3.7. (*Q* form of The Girard-Newton Identities, by Chen Shuwen, 2017-2019)

Let n and k be positive integers. Denote

$$P_k = a_1^k + a_2^k + \cdots + a_n^k$$

and

$$Q_k = -P_k/k$$

and

$$S_k = \sum_{\substack{0 \leq i_1, i_2, i_3, \dots, i_k \leq k \\ i_1 + 2i_2 + \cdots + ki_k = k}} \frac{Q_1^{i_1} Q_2^{i_2} \cdots Q_k^{i_k}}{i_1! i_2! \cdots i_k!}$$

then

$$S_k = (-1)^k \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} a_{j_1} a_{j_2} \cdots a_{j_k} \quad k \leq n$$

$$S_k = 0 \quad k > n$$

Identities 3.8. (*F* form of The Girard-Newton Identities, by Chen Shuwen, 2017-2019)

Let n and k be positive integers. Denote

$$P_k = a_1^k + a_2^k + \cdots + a_n^k$$

and

$$F_1 = -P_1$$

$$F_k = (P_1^k - P_k)/k \quad k > 1$$

and

$$S_k = \sum_{\substack{0 \leq i_1 \leq 1 \\ 0 \leq i_2, i_3, \dots, i_k \leq k \\ i_1 + 2i_2 + \cdots + ki_k = k}} \frac{F_1^{i_1} F_2^{i_2} F_3^{i_3} \cdots F_k^{i_k}}{i_2! i_3! \cdots i_k!}$$

then

$$S_k = (-1)^k \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} a_{j_1} a_{j_2} \cdots a_{j_k} \quad k \leq n$$

$$S_k = 0 \quad k > n$$

Identities 3.9. (*D* form of The Girard-Newton Identities, by Chen Shuwen, 2020)

Let n and k be positive integers. Denote

$$P_k = a_1^k + a_2^k + \cdots + a_n^k$$

and

$$D_1 = -P_1$$

$$D_k = (P_1^k - P_k)/k$$

k is prime number

$$D_k = (P_1^k - P_k)/k + \sum_{j=1}^h (-1)^{f_j} \frac{1}{f_j} D_{k/f_j}^{f_j}$$

k is composite number with all its factors

f_1, f_2, \dots, f_h , exclude 1 and itself.

and

$$S_k = \sum_{\substack{0 \leq i_1, i_2, i_3, \dots, i_k \leq 1 \\ i_1 + 2i_2 + \dots + ki_k = k}} D_1^{i_1} D_2^{i_2} D_3^{i_3} \dots D_k^{i_k}$$

then

$$S_k = (-1)^k \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} a_{j_1} a_{j_2} \dots a_{j_k} \quad k \leq n$$

$$S_k = 0 \quad k > n$$

Example 3.4.

Denote

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k \quad (k = 1, 2, 3, 4, 5, 6, 7)$$

then

(1) By Identities 3.6, we have

$$S_7 = -\frac{P_1^7}{5040} + \frac{1}{240} P_1^5 P_2 - \frac{1}{48} P_1^3 P_2^2 + \frac{1}{48} P_1 P_2^3 - \frac{1}{72} P_1^4 P_3 + \frac{1}{12} P_1^2 P_2 P_3 - \frac{1}{24} P_2^2 P_3 - \frac{1}{18} P_1 P_3^2 + \frac{1}{24} P_1^3 P_4 - \frac{1}{8} P_1 P_2 P_4 + \frac{P_3 P_4}{12} - \frac{1}{10} P_1^2 P_5 + \frac{P_2 P_5}{10} + \frac{P_1 P_6}{6} - \frac{P_7}{7} = 0$$

(2) By Identities 3.7, we have

$$S_7 = \frac{Q_1^7}{7!} + \frac{Q_1^5}{5!} Q_2 + \frac{Q_1^3}{3!} \frac{Q_2^2}{2!} + Q_1 \frac{Q_2^3}{3!} + \frac{Q_1^4}{4!} Q_3 + \frac{Q_1^2}{2!} Q_2 Q_3 + \frac{Q_2^2}{2!} Q_3 + Q_1 \frac{Q_3^2}{2!} + \frac{Q_1^3}{3!} Q_4 + Q_1 Q_2 Q_4 + Q_3 Q_4 + \frac{Q_1^2}{2!} Q_5 + Q_2 Q_5 + Q_1 Q_6 + Q_7 = 0$$

$$\text{here } Q_k = -P_k/k \quad (k = 1, 2, 3, 4, 5, 6, 7)$$

(3) By Identities 3.8, we have

$$S_7 = \frac{F_2^3}{3!} F_1 + \frac{F_2^2}{2!} F_3 + \frac{F_3^2}{2!} F_1 + F_1 F_2 F_4 + F_3 F_4 + F_2 F_5 + F_1 F_6 + F_7 = 0$$

$$\text{here } F_1 = -P_1, F_k = (P_1^k - P_k)/k \quad (k = 2, 3, 4, 5, 6, 7)$$

(4) By Identities 3.9, we have

$$S_7 = D_1 D_2 D_4 + D_3 D_4 + D_2 D_5 + D_1 D_6 + D_7 = 0$$

here $D_1 = -P_1$ and

$$D_k = (P_1^k - P_k)/k \quad (k = 2, 3, 5, 7)$$

$$D_4 = (P_1^4 - P_4)/4 + \frac{D_2^2}{2} \quad \left(= \frac{3P_1^4}{8} - \frac{1}{4} P_1^2 P_2 + \frac{P_2^2}{8} - \frac{P_4}{4} \right)$$

$$D_6 = (P_1^6 - P_6)/4 - \frac{D_2^3}{3} + \frac{D_3^2}{2} \quad \left(= \frac{13P_1^6}{72} + \frac{1}{8} P_1^4 P_2 - \frac{1}{8} P_1^2 P_2^2 + \frac{P_2^3}{24} - \frac{1}{9} P_1^3 P_3 + \frac{P_3^2}{18} - \frac{P_6}{6} \right)$$

Corollary 3.1. (Proof by Identities 3.6 or 3.7 or 3.8 or 3.9)

1. There is no non-trivial solution for

$$a_1^k + a_2^k + a_3^k + \dots + a_n^k = b_1^k + b_2^k + b_3^k + \dots + b_n^k \quad (k = 1, 2, 3, \dots, n)$$

2. If

$$a_1^k + a_2^k + a_3^k + \dots + a_n^k = b_1^k + b_2^k + b_3^k + \dots + b_n^k \quad (k = 1, 2, 3, \dots, n - 1, n + 1)$$

then

$$a_1 + a_2 + a_3 + \dots + a_n = b_1 + b_2 + b_3 + \dots + b_n = 0$$

Identities 3.10. (A generalization of the Girard-Newton Identities, by Chen Shuwen, 2017)

Let

$$\begin{cases} P_k = a_1^k + a_2^k + \dots + a_n^k, & \text{for } k \neq 0 \\ P_0 = a_1 a_2 \dots a_n \end{cases}$$

and

$$\begin{cases} S_1 = -P_1 \\ S_k = -(P_k + S_1 P_{k-1} + S_2 P_{k-2} + \dots + S_{k-1} P_1)/k, & \text{for } k > 1 \\ S_0 = -P_0 \\ S_{-1} = P_0 P_{-1} \\ S_k = -(-P_0 P_k + S_{-1} P_{k+1} + S_{-2} P_{k+2} + \dots + S_{k+1} P_{-1})/(-k), & \text{for } k < -1 \end{cases}$$

and

$$\begin{cases} T_k = (-1)^n S_{k-n}, & \text{for } k \leq -1 \\ T_0 = 1 + (-1)^n S_{-n} \\ T_k = S_k + (-1)^n S_{k-n}, & \text{for } 1 \leq k \leq n \\ T_k = S_k, & \text{for } k \geq n + 1 \end{cases}$$

then

$$T_k = 0$$

Remark. Here we generalize The Girard-Newton Identities from the case of $k > 0$ to the case of any integer k . It is useful for solving the system of the below form

$$\begin{aligned} a_1^k + a_2^k + a_3^k + \dots + a_n^k &= b_1^k + b_2^k + b_3^k + \dots + b_n^k & (k = k_1, k_2, k_3, \dots, k_m) \\ a_1 a_2 \dots a_n &= b_1 b_2 \dots b_n \end{aligned}$$

We will discuss this system in Chapter 5.

Example (Chen Shuwen, 2019/2/5).

$$\begin{aligned} \frac{1}{-63} + \frac{1}{-45} + \frac{1}{8} + \frac{1}{16} + \frac{1}{84} &= \frac{1}{-72} + \frac{1}{-36} + \frac{1}{7} + \frac{1}{21} + \frac{1}{80} \\ (-63) \times (-45) \times 8 \times 16 \times 84 &= (-72) \times (-36) \times 7 \times 21 \times 80 \\ (-63) + (-45) + 8 + 16 + 84 &= (-72) + (-36) + 7 + 21 + 80 \\ (-63)^2 + (-45)^2 + 8^2 + 16^2 + 84^2 &= (-72)^2 + (-36)^2 + 7^2 + 21^2 + 80^2 \\ (-63)^4 + (-45)^4 + 8^4 + 16^4 + 84^4 &= (-72)^4 + (-36)^4 + 7^4 + 21^4 + 80^4 \end{aligned}$$

Identities 3.11. (A generalization of the Girard-Newton Identities, by Chen Shuwen, 1997, 2016, 2020)

Let n and k and be positive integers. Denote

$$P_{2k+1} = a_1^{2k+1} + a_2^{2k+1} + \dots + a_n^{2k+1}$$

and

$$G_1 = -P_1$$

$$G_3 = (P_1^3 - P_3)/3$$

$$G_5 = (P_1^5 - P_5)/5$$

$$G_7 = (P_1^7 - P_7)/7$$

$$G_9 = (P_1^9 - P_9)/9 - \left(\frac{G_3^3}{3}\right)$$

$$G_{11} = (P_1^{11} - P_{11})/11 - (G_3^2 G_5)$$

$$G_{13} = (P_1^{13} - P_{13})/13 - (G_3 G_5^2 + G_3^2 G_7)$$

$$G_{15} = (P_1^{15} - P_{15})/15 - \left(\frac{G_3^5}{5} + \frac{G_5^3}{3} + 2G_3 G_5 G_7 + G_3^2 G_9\right)$$

$$G_{17} = (P_1^{17} - P_{17})/17 - (G_3^4 G_5 + G_5^2 G_7 + G_3 G_7^2 + 2G_3 G_5 G_9 + G_3^2 G_{11})$$

⋮

In general

$$G_{2k+1} =$$

where

$$Z_{2k} =$$

Let

$$X_{n,0} = \begin{cases} (-1)^{\frac{n(n+2)}{8}} \begin{vmatrix} G_{n-1} & G_{n-3} & G_{n-5} & \dots & G_5 & G_3 & & & G_1 \\ G_{n+1} & G_{n-1} & G_{n-3} & \dots & G_7 & G_5 & & & G_1^3 + G_3 \\ G_{n+3} & G_{n+1} & G_{n-1} & \dots & G_9 & G_7 & & & G_1^5 + G_3 G_1^2 + G_5 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ G_{2n-3} & G_{2n-5} & G_{2n-7} & \dots & G_{n+3} & G_{n+1} & G_1^{n-1} + G_3 G_1^{n-4} + \dots + G_{n-3} G_1^2 + G_{n-1} & & \end{vmatrix}, & n \text{ is even} \\ (-1)^{\frac{n^2-1}{8}} \begin{vmatrix} G_{n-2} & G_{n-4} & G_{n-6} & \dots & G_3 & 0 & & & 1 \\ G_n & G_{n-2} & G_{n-4} & \dots & G_5 & G_3 & & & G_1^2 \\ G_{n+2} & G_n & G_{n-2} & \dots & G_7 & G_5 & & & G_1^4 + G_3 G_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ G_{2n-3} & G_{2n-5} & G_{2n-7} & \dots & G_{n+2} & G_n & G_1^{n-1} + G_3 G_1^{n-4} + \dots + G_{n-4} G_1^3 + G_{n-2} G_1 & & \end{vmatrix}, & n \text{ is odd} \end{cases}$$

$X_{n,2}$

$$= \begin{cases} (-1)^{\frac{n(n+2)}{8}+1} \begin{vmatrix} G_{n+1} & G_{n-3} & G_{n-5} & \dots & G_5 & G_3 & & & G_1 \\ G_{n+3} & G_{n-1} & G_{n-3} & \dots & G_7 & G_5 & & & G_1^3 + G_3 \\ G_{n+5} & G_{n+1} & G_{n-1} & \dots & G_9 & G_7 & & & G_1^5 + G_3 G_1^2 + G_5 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ G_{2n-1} & G_{2n-5} & G_{2n-7} & \dots & G_{n+3} & G_{n+1} & G_1^{n-1} + G_3 G_1^{n-4} + \dots + G_{n-3} G_1^2 + G_{n-1} & & \end{vmatrix}, & n \text{ is even} \\ (-1)^{\frac{n^2-1}{8}+1} \begin{vmatrix} G_n & G_{n-4} & G_{n-6} & \dots & G_3 & 0 & & & 1 \\ G_{n+2} & G_{n-2} & G_{n-4} & \dots & G_5 & G_3 & & & G_1^2 \\ G_{n+4} & G_n & G_{n-2} & \dots & G_7 & G_5 & & & G_1^4 + G_3 G_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ G_{2n-1} & G_{2n-5} & G_{2n-7} & \dots & G_{n+2} & G_n & G_1^{n-1} + G_3 G_1^{n-4} + \dots + G_{n-4} G_1^3 + G_{n-2} G_1 & & \end{vmatrix}, & n \text{ is odd} \end{cases}$$

$$X_{n,n} = \begin{cases} (-1)^{\frac{n(n-2)}{8}} \begin{vmatrix} G_{n+1} & G_{n-1} & G_{n-3} & \dots & G_5 & G_3 \\ G_{n+3} & G_{n+1} & G_{n-1} & \dots & G_7 & G_5 \\ G_{n+5} & G_{n+3} & G_{n+1} & \dots & G_9 & G_7 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ G_{2n-1} & G_{2n-3} & G_{2n-5} & \dots & G_{n+3} & G_{n+1} \end{vmatrix}, & n \text{ is even} \\ (-1)^{\frac{(n-2)^2-1}{8}} \begin{vmatrix} G_n & G_{n-2} & G_{n-4} & \dots & G_3 & 0 \\ G_{n+2} & G_n & G_{n-2} & \dots & G_5 & G_3 \\ G_{n+4} & G_{n+2} & G_n & \dots & G_7 & G_5 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ G_{2n-1} & G_{2n-3} & G_{2n-5} & \dots & G_{n+2} & G_n \end{vmatrix}, & n \text{ is odd} \end{cases}$$

$U_{1,3,5,\dots,2n-3,2n+1}$

$$= \begin{cases} (-1)^{\frac{n(n+2)}{8}+1} \begin{vmatrix} G_{n+1} & G_{n-3} & G_{n-5} & \dots & G_5 & G_3 & G_1 \\ G_{n+3} & G_{n-1} & G_{n-3} & \dots & G_7 & G_5 & G_1^3 + G_3 \\ G_{n+5} & G_{n+1} & G_{n-1} & \dots & G_9 & G_7 & G_1^5 + G_3G_1^2 + G_5 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ G_{2n-3} & G_{2n-7} & G_{2n-9} & \dots & G_{n+1} & G_{n-1} & G_1^{n-3} + G_3G_1^{n-6} + \dots + G_{n-5}G_1^2 + G_{n-3} \\ G_{2n+1} & G_{2n-3} & G_{2n-5} & \dots & G_{n+5} & G_{n+3} & G_1^{n+1} + G_3G_1^{n-2} + \dots + G_{n-1}G_1^2 + G_{n+1} \end{vmatrix}, & n \text{ is even} \\ (-1)^{\frac{n^2-1}{8}+1} \begin{vmatrix} G_n & G_{n-4} & G_{n-6} & \dots & G_3 & 0 & 1 \\ G_{n+2} & G_{n-2} & G_{n-4} & \dots & G_5 & G_3 & G_1^2 \\ G_{n+4} & G_n & G_{n-2} & \dots & G_7 & G_5 & G_1^4 + G_3G_1 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ G_{2n-3} & G_{2n-7} & G_{2n-9} & \dots & G_n & G_{n-2} & G_1^{n-3} + G_3G_1^{n-6} + \dots + G_{n-6}G_1^3 + G_{n-4}G_1 \\ G_{2n+1} & G_{2n-3} & G_{2n-5} & \dots & G_{n+4} & G_{n+2} & G_1^{n+1} + G_3G_1^{n-2} + \dots + G_{n-2}G_1^3 + G_nG_1 \end{vmatrix}, & n \text{ is odd} \end{cases}$$

then

$$X_{n,0} = \prod_{1 \leq i < j \leq n} (a_i + a_j)$$

$$X_{n,2} = \prod_{1 \leq i < j \leq n} (a_i + a_j) \sum_{1 \leq i < j \leq n} a_i a_j$$

$$X_{n,n} = \prod_{1 \leq i < j \leq n} (a_i + a_j) \prod_{1 \leq i \leq n} (a_1 + a_2 + \dots + a_n - a_i)$$

$$U_{1,3,5,\dots,2n-3,2n+1} = \prod_{1 \leq i < j \leq n} (a_i + a_j) \sum_{1 \leq i < j \leq n} a_i a_j \left(\sum_{1 \leq i < j \leq n} a_i a_j + \sum_{1 \leq i \leq n} a_i^2 \right)$$

and then

$$X_{n,2}/X_{n,0} = \sum_{1 \leq i < j \leq n} a_i a_j$$

$$X_{n,n}/X_{n,0} = \prod_{1 \leq i \leq n} (a_1 + a_2 + \dots + a_n - a_i)$$

$$G_1^4 - 4 U_{1,3,5,\dots,2n-3,2n+1}/X_{n,0} = \left(\sum_{1 \leq i \leq n} a_i^2 \right)^2$$

Remark. Here we generalize The Girard-Newton Identities from the case of positive integer k to the case of all k be positive odd integers. It is useful for solving the system of the below form

$$a_1^k + a_2^k + a_3^k + \dots + a_n^k = b_1^k + b_2^k + b_3^k + \dots + b_m^k \quad (k = 2k_1 + 1, 2k_2 + 1, \dots, 2k_m + 1)$$

In the next section, we will generalize it to the case of all k be odd integers (both positive and negative).

Example 3.5. (Chen Shuwen, 2019, 2020)

Let n and k and be positive integers. Denote

$$P_{2k+1} = a_1^{2k+1} + a_2^{2k+1} + \cdots + a_n^{2k+1}$$

and

$$G_1 = -P_1$$

$$G_3 = (P_1^3 - P_3)/3$$

$$G_5 = (P_1^5 - P_5)/5$$

$$G_7 = (P_1^7 - P_7)/7$$

$$G_9 = (P_1^9 - P_9)/9 - \left(\frac{G_3^3}{3}\right)$$

$$G_{11} = (P_1^{11} - P_{11})/11 - (G_3^2 G_5)$$

$$G_{13} = (P_1^{13} - P_{13})/13 - (G_3 G_5^2 + G_3^2 G_7)$$

$$G_{15} = (P_1^{15} - P_{15})/15 - \left(\frac{G_3^5}{5} + \frac{G_5^3}{3} + 2G_3 G_5 G_7 + G_3^2 G_9\right)$$

$$G_{17} = (P_1^{17} - P_{17})/17 - (G_3^4 G_5 + G_5^2 G_7 + G_3 G_7^2 + 2G_3 G_5 G_9 + G_3^2 G_{11})$$

$$G_{19} = (P_1^{19} - P_{19})/19 - (2G_3^3 G_5^2 + G_3^4 G_7 + G_5 G_7^2 + G_5^2 G_9 + 2G_3 G_7 G_9 + 2G_3 G_5 G_{11} + G_3^2 G_{13})$$

then

$$X_{2,0} = -|G_1| = \prod_{1 \leq i < j \leq 2} (a_i + a_j) = (a_1 + a_2)$$

$$X_{3,0} = -\begin{vmatrix} 0 & 1 \\ G_3 & G_1^2 \end{vmatrix} = \prod_{1 \leq i < j \leq 3} (a_i + a_j) = (a_1 + a_2)(a_1 + a_3)(a_2 + a_3)$$

$$X_{4,0} = -\begin{vmatrix} G_3 & G_1 \\ G_5 & G_1^3 + G_3 \end{vmatrix} = \prod_{1 \leq i < j \leq 4} (a_i + a_j) = (a_1 + a_2)(a_1 + a_3)(a_2 + a_3)(a_1 + a_4)(a_2 + a_4)(a_3 + a_4)$$

$$X_{5,0} = -\begin{vmatrix} G_3 & 0 & 1 \\ G_5 & G_3 & G_1^2 \\ G_7 & G_5 & G_1^4 + G_3 G_1 \end{vmatrix} = \prod_{1 \leq i < j \leq 5} (a_i + a_j)$$

$$X_{6,0} = \begin{vmatrix} G_5 & G_3 & G_1 \\ G_7 & G_5 & G_1^3 + G_3 \\ G_9 & G_7 & G_1^5 + G_3 G_1^2 + G_5 \end{vmatrix} = \prod_{1 \leq i < j \leq 6} (a_i + a_j)$$

$$X_{7,0} = \begin{vmatrix} G_5 & G_3 & 0 & 1 \\ G_7 & G_5 & G_3 & G_1^2 \\ G_9 & G_7 & G_5 & G_1^4 + G_3 G_1 \\ G_{11} & G_9 & G_7 & G_1^6 + G_3 G_1^3 + G_5 G_1 \end{vmatrix} = \prod_{1 \leq i < j \leq 7} (a_i + a_j)$$

$$X_{8,0} = \begin{vmatrix} G_7 & G_5 & G_3 & G_1 \\ G_9 & G_7 & G_5 & G_1^3 + G_3 \\ G_{11} & G_9 & G_7 & G_1^5 + G_3 G_1^2 + G_5 \\ G_{13} & G_{11} & G_9 & G_1^7 + G_3 G_1^4 + G_5 G_1^2 + G_7 \end{vmatrix} = \prod_{1 \leq i < j \leq 8} (a_i + a_j)$$

$$X_{9,0} = \begin{vmatrix} G_7 & G_5 & G_3 & 0 & 1 \\ G_9 & G_7 & G_5 & G_3 & G_1^2 \\ G_{11} & G_9 & G_7 & G_5 & G_1^4 + G_3 G_1 \\ G_{13} & G_{11} & G_9 & G_7 & G_1^6 + G_3 G_1^3 + G_5 G_1 \\ G_{15} & G_{13} & G_{11} & G_9 & G_1^8 + G_3 G_1^5 + G_5 G_1^3 + G_7 G_1 \end{vmatrix} = \prod_{1 \leq i < j \leq 9} (a_i + a_j)$$

$$X_{10,0} = - \begin{vmatrix} G_9 & G_7 & G_5 & G_3 & & G_1 \\ G_{11} & G_9 & G_7 & G_5 & & G_1^3 + G_3 \\ G_{13} & G_{11} & G_9 & G_7 & & G_1^5 + G_3G_1^2 + G_5 \\ G_{15} & G_{13} & G_{11} & G_9 & & G_1^7 + G_3G_1^4 + G_5G_1^2 + G_7 \\ G_{17} & G_{15} & G_{13} & G_{11} & & G_1^9 + G_3G_1^6 + G_5G_1^4 + G_7G_1^2 + G_9 \end{vmatrix} = \prod_{1 \leq i < j \leq 10} (a_i + a_j)$$

$$X_{11,0} = - \begin{vmatrix} G_9 & G_7 & G_5 & G_3 & 0 & & 1 \\ G_{11} & G_9 & G_7 & G_5 & G_3 & & G_1^2 \\ G_{13} & G_{11} & G_9 & G_7 & G_5 & & G_1^4 + G_3G_1 \\ G_{15} & G_{13} & G_{11} & G_9 & G_7 & & G_1^6 + G_3G_1^3 + G_5G_1 \\ G_{17} & G_{15} & G_{13} & G_{11} & G_9 & & G_1^8 + G_3G_1^5 + G_5G_1^3 + G_7G_1 \\ G_{19} & G_{17} & G_{15} & G_{13} & G_{11} & & G_1^{10} + G_3G_1^7 + G_5G_1^5 + G_7G_1^3 + G_9G_1 \end{vmatrix} = \prod_{1 \leq i < j \leq 11} (a_i + a_j)$$

$$X_{12,0} = - \begin{vmatrix} G_{11} & G_9 & G_7 & G_5 & G_3 & & & G_1 \\ G_{13} & G_{11} & G_9 & G_7 & G_5 & & & G_1^3 + G_3 \\ G_{15} & G_{13} & G_{11} & G_9 & G_7 & & & G_1^5 + G_3G_1^2 + G_5 \\ G_{17} & G_{15} & G_{13} & G_{11} & G_9 & & & G_1^7 + G_3G_1^4 + G_5G_1^2 + G_7 \\ G_{19} & G_{17} & G_{15} & G_{13} & G_{11} & & & G_1^9 + G_3G_1^6 + G_5G_1^4 + G_7G_1^2 + G_9 \\ G_{21} & G_{19} & G_{17} & G_{15} & G_{13} & & & G_1^{11} + G_3G_1^8 + G_5G_1^6 + G_7G_1^4 + G_9G_1^2 + G_{11} \end{vmatrix} = \prod_{1 \leq i < j \leq 12} (a_i + a_j)$$

$$X_{13,0} = - \begin{vmatrix} G_{11} & G_9 & G_7 & G_5 & G_3 & 0 & & & 1 \\ G_{13} & G_{11} & G_9 & G_7 & G_5 & G_3 & & & G_1^2 \\ G_{15} & G_{13} & G_{11} & G_9 & G_7 & G_5 & & & G_1^4 + G_3G_1 \\ G_{17} & G_{15} & G_{13} & G_{11} & G_9 & G_7 & & & G_1^6 + G_3G_1^3 + G_5G_1 \\ G_{19} & G_{17} & G_{15} & G_{13} & G_{11} & G_9 & & & G_1^8 + G_3G_1^5 + G_5G_1^3 + G_7G_1 \\ G_{21} & G_{19} & G_{17} & G_{15} & G_{13} & G_{11} & & & G_1^{10} + G_3G_1^7 + G_5G_1^5 + G_7G_1^3 + G_9G_1 \\ G_{23} & G_{21} & G_{19} & G_{17} & G_{15} & G_{13} & & & G_1^{12} + G_3G_1^9 + G_5G_1^7 + G_7G_1^5 + G_9G_1^3 + G_{11}G_1 \end{vmatrix} = \prod_{1 \leq i < j \leq 13} (a_i + a_j)$$

$$X_{14,0} = \begin{vmatrix} G_{13} & G_{11} & G_9 & G_7 & G_5 & G_3 & & & & G_1 \\ G_{15} & G_{13} & G_{11} & G_9 & G_7 & G_5 & & & & G_1^3 + G_3 \\ G_{17} & G_{15} & G_{13} & G_{11} & G_9 & G_7 & & & & G_1^5 + G_3G_1^2 + G_5 \\ G_{19} & G_{17} & G_{15} & G_{13} & G_{11} & G_9 & & & & G_1^7 + G_3G_1^4 + G_5G_1^2 + G_7 \\ G_{21} & G_{19} & G_{17} & G_{15} & G_{13} & G_{11} & & & & G_1^9 + G_3G_1^6 + G_5G_1^4 + G_7G_1^2 + G_9 \\ G_{23} & G_{21} & G_{19} & G_{17} & G_{15} & G_{13} & & & & G_1^{11} + G_3G_1^8 + G_5G_1^6 + G_7G_1^4 + G_9G_1^2 + G_{11} \\ G_{25} & G_{23} & G_{21} & G_{19} & G_{17} & G_{15} & & & & G_1^{13} + G_3G_1^{10} + G_5G_1^8 + G_7G_1^6 + G_9G_1^4 + G_{11}G_1^2 + G_{13} \end{vmatrix} = \prod_{1 \leq i < j \leq 14} (a_i + a_j)$$

$$X_{15,0} = \begin{vmatrix} G_{13} & G_{11} & G_9 & G_7 & G_5 & G_3 & 0 & & & & 1 \\ G_{15} & G_{13} & G_{11} & G_9 & G_7 & G_5 & G_3 & & & & G_1^2 \\ G_{17} & G_{15} & G_{13} & G_{11} & G_9 & G_7 & G_5 & & & & G_1^4 + G_3G_1 \\ G_{19} & G_{17} & G_{15} & G_{13} & G_{11} & G_9 & G_7 & & & & G_1^6 + G_3G_1^3 + G_5G_1 \\ G_{21} & G_{19} & G_{17} & G_{15} & G_{13} & G_{11} & G_9 & & & & G_1^8 + G_3G_1^5 + G_5G_1^3 + G_7G_1 \\ G_{23} & G_{21} & G_{19} & G_{17} & G_{15} & G_{13} & G_{11} & & & & G_1^{10} + G_3G_1^7 + G_5G_1^5 + G_7G_1^3 + G_9G_1 \\ G_{25} & G_{23} & G_{21} & G_{19} & G_{17} & G_{15} & G_{13} & & & & G_1^{12} + G_3G_1^9 + G_5G_1^7 + G_7G_1^5 + G_9G_1^3 + G_{11}G_1 \\ G_{27} & G_{25} & G_{23} & G_{21} & G_{19} & G_{17} & G_{15} & & & & G_1^{14} + G_3G_1^{11} + G_5G_1^9 + G_7G_1^7 + G_9G_1^5 + G_{11}G_1^3 + G_{13}G_1 \end{vmatrix} = \prod_{1 \leq i < j \leq 15} (a_i + a_j)$$

Example 3.6. (Chen Shuwen, 2020)

<p>Let</p> $P_k = a_1^k + a_2^k \quad (k = 1, 3)$ <p>and</p> $G_1 = -P_1$ $G_3 = (P_1^3 - P_3)/3$ <p>then</p> $G_1^2 + 2 \frac{ G_3 }{ G_1 } = a_1^2 + a_2^2$	<p>Let</p> $P_k = a_1^k + a_2^k \quad (k = 1, 5)$ <p>and</p> $G_1 = -P_1$ $G_5 = (P_1^5 - P_5)/5$ <p>then</p> $G_1^4 + 4 \frac{ G_5 }{ G_1 } = (a_1^2 + a_2^2)^2$
<p>Let</p> $P_k = a_1^k + a_2^k + a_3^k \quad (k = 1, 3, 5)$ <p>and</p> $G_1 = -P_1$ $G_3 = (P_1^3 - P_3)/3$ $G_5 = (P_1^5 - P_5)/5$ <p>then</p> $G_1^2 + 2 \frac{\begin{vmatrix} G_3 & 1 \\ G_5 & G_1^2 \end{vmatrix}}{\begin{vmatrix} 0 & 1 \\ G_3 & G_1^2 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2$	<p>Let</p> $P_k = a_1^k + a_2^k + a_3^k \quad (k = 1, 3, 7)$ <p>and</p> $G_1 = -P_1$ $G_3 = (P_1^3 - P_3)/3$ $G_7 = (P_1^7 - P_7)/7$ <p>then</p> $G_1^4 + 4 \frac{\begin{vmatrix} G_3 & 1 \\ G_7 & G_1^4 + G_1 G_3 \end{vmatrix}}{\begin{vmatrix} 0 & 1 \\ G_3 & G_1^2 \end{vmatrix}} = (a_1^2 + a_2^2 + a_3^2)^2$
<p>Let</p> $P_k = a_1^k + a_2^k + a_3^k + a_4^k \quad (k = 1, 3, 5, 7)$ <p>and</p> $G_1 = -P_1$ $G_3 = (P_1^3 - P_3)/3$ $G_5 = (P_1^5 - P_5)/5$ $G_7 = (P_1^7 - P_7)/7$ <p>then</p> $G_1^2 + 2 \frac{\begin{vmatrix} G_5 & G_1 \\ G_7 & G_1^3 + G_3 \end{vmatrix}}{\begin{vmatrix} G_3 & G_1 \\ G_5 & G_1^3 + G_3 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + a_4^2$	<p>Let</p> $P_k = a_1^k + a_2^k + a_3^k + a_4^k \quad (k = 1, 3, 5, 9)$ <p>and</p> $G_1 = -P_1$ $G_3 = (P_1^3 - P_3)/3$ $G_5 = (P_1^5 - P_5)/5$ $G_9 = (P_1^9 - P_9)/9 - (G_3^3/3)$ <p>then</p> $G_1^4 + 4 \frac{\begin{vmatrix} G_5 & G_1 \\ G_9 & G_1^5 + G_1^2 G_3 + G_5 \end{vmatrix}}{\begin{vmatrix} G_3 & G_1 \\ G_5 & G_1^3 + G_3 \end{vmatrix}} = (a_1^2 + a_2^2 + a_3^2 + a_4^2)^2$
<p>Let</p> $P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k \quad (k = 1, 3, 5, 7, 9)$ <p>and</p> $G_1 = -P_1$ $G_3 = (P_1^3 - P_3)/3$ $G_5 = (P_1^5 - P_5)/5$ $G_7 = (P_1^7 - P_7)/7$ $G_9 = (P_1^9 - P_9)/9 - (G_3^3/3)$ <p>then</p> $G_1^2 + 2 \frac{\begin{vmatrix} G_5 & 0 & 1 \\ G_7 & G_3 & G_1^2 \\ G_9 & G_5 & G_1^4 + G_3 G_1 \end{vmatrix}}{\begin{vmatrix} G_3 & 0 & 1 \\ G_5 & G_3 & G_1^2 \\ G_7 & G_5 & G_1^4 + G_3 G_1 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2$	<p>Let</p> $P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k \quad (k = 1, 3, 5, 7, 11)$ <p>and</p> $G_1 = -P_1$ $G_3 = (P_1^3 - P_3)/3$ $G_5 = (P_1^5 - P_5)/5$ $G_7 = (P_1^7 - P_7)/7$ $G_{11} = (P_1^{11} - P_{11})/11 - (G_3^2 G_5)$ <p>then</p> $G_1^4 + 4 \frac{\begin{vmatrix} G_5 & 0 & 1 \\ G_7 & G_3 & G_1^2 \\ G_{11} & G_7 & G_1^6 + G_3 G_1^3 + G_5 G_1 \end{vmatrix}}{\begin{vmatrix} G_3 & 0 & 1 \\ G_5 & G_3 & G_1^2 \\ G_7 & G_5 & G_1^4 + G_3 G_1 \end{vmatrix}} = (a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2)^2$

Corollary 3.2 (Chen Shuwen, 2020)

If

$$\sum_{i=1}^n a_i^k = \sum_{i=1}^n b_i^k \quad (k = 1, 3, 5, \dots, 2n - 1)$$

then

$$\prod_{1 \leq i < j \leq n} (a_i + a_j) = \prod_{1 \leq i < j \leq n} (b_i + b_j)$$

Example 3.7.1 (Chen Shuwen, 1999, 2020)

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k \quad (k = 1, 3, 5, 7, 9, 11)$$

and

$$G_1 = -P_1$$

$$G_3 = (P_1^3 - P_3)/3$$

$$G_5 = (P_1^5 - P_5)/5$$

$$G_7 = (P_1^7 - P_7)/7$$

$$G_9 = (P_1^9 - P_9)/9 - (G_3^3/3)$$

$$G_{11} = (P_1^{11} - P_{11})/11 - (G_3^2 G_5)$$

then

$$G_1^2 + 2 \frac{\begin{vmatrix} G_7 & G_3 & G_1 \\ G_9 & G_5 & G_1^3 + G_3 \\ G_{11} & G_7 & G_1^5 + G_3 G_1^2 + G_5 \end{vmatrix}}{\begin{vmatrix} G_5 & G_3 & G_1 \\ G_7 & G_5 & G_1^3 + G_3 \\ G_9 & G_7 & G_1^5 + G_3 G_1^2 + G_5 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2$$

Example 3.7.2 (Chen Shuwen, 2020)

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k \quad (k = 1, 3, 5, 7, 9, 13)$$

and

$$G_1 = -P_1$$

$$G_3 = (P_1^3 - P_3)/3$$

$$G_5 = (P_1^5 - P_5)/5$$

$$G_7 = (P_1^7 - P_7)/7$$

$$G_9 = (P_1^9 - P_9)/9 - (G_3^3/3)$$

$$G_{13} = (P_1^{13} - P_{13})/13 - (G_3 G_5^2 + G_3^2 G_7)$$

then

$$G_1^4 + 4 \frac{\begin{vmatrix} G_7 & G_3 & G_1 \\ G_9 & G_5 & G_1^3 + G_3 \\ G_{13} & G_7 & G_1^7 + G_3 G_1^4 + G_5 G_1^2 + G_7 \end{vmatrix}}{\begin{vmatrix} G_5 & G_3 & G_1 \\ G_7 & G_5 & G_1^3 + G_3 \\ G_9 & G_7 & G_1^5 + G_3 G_1^2 + G_5 \end{vmatrix}} = (a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2)^2$$

Example 3.7.3 (Chen Shuwen, 2020)

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k \quad (k = -1, 1, 3, 5, 7, 9)$$

and

$$G_{-1} = P_{-1}$$

$$G_1 = -P_1$$

$$G_3 = (P_1^3 - P_3)/3$$

$$G_5 = (P_1^5 - P_5)/5$$

$$G_7 = (P_1^7 - P_7)/7$$

$$G_9 = (P_1^9 - P_9)/9 - (G_3^3/3)$$

then

$$G_1^2 + 2 \frac{\begin{vmatrix} G_5(1 + G_{-1}G_1) & G_1 & G_{-1} \\ G_7 & G_3 & G_1 \\ G_9 & G_5 & G_1^3 + G_3 \end{vmatrix}}{\begin{vmatrix} G_3(1 + G_{-1}G_1) & G_1 & G_{-1} \\ G_5 & G_3 & G_1 \\ G_7 & G_5 & G_1^3 + G_3 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2$$

Example 3.7.4 (Chen Shuwen, 2020)

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k \quad (k = -1, 1, 3, 5, 7, 11)$$

and

$$G_{-1} = P_{-1}$$

$$G_1 = -P_1$$

$$G_3 = (P_1^3 - P_3)/3$$

$$G_5 = (P_1^5 - P_5)/5$$

$$G_7 = (P_1^7 - P_7)/7$$

$$G_{11} = (P_1^{11} - P_{11})/11 - (G_3^2 G_5)$$

then

$$G_1^4 + 4 \frac{\begin{vmatrix} G_5(1 + G_{-1}G_1) & G_1 & G_{-1} \\ G_7 & G_3 & G_1 \\ G_{11} & G_7 & G_1^5 + G_3 G_1^2 + G_5 \end{vmatrix}}{\begin{vmatrix} G_3(1 + G_{-1}G_1) & G_1 & G_{-1} \\ G_5 & G_3 & G_1 \\ G_7 & G_5 & G_1^3 + G_3 \end{vmatrix}} = (a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2)^2$$

Example 3.7.5 (Chen Shuwen, 2020)

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k \quad (k = -3, -1, 1, 3, 5, 7)$$

and

$$G_{-1} = P_{-1}$$

$$G_{-3} = -(P_{-1}^3 - P_{-3})/3$$

$$G_1 = -P_1$$

$$G_3 = (P_1^3 - P_3)/3$$

$$G_5 = (P_1^5 - P_5)/5$$

$$G_7 = (P_1^7 - P_7)/7$$

then

$$G_1^2 + 2 \frac{\begin{vmatrix} G_3(1 + G_{-1}G_1) & G_{-1} & G_{-3} \\ G_5 & G_1 & G_{-1} \\ G_7 - G_1^2G_5 & G_3 & G_1 \end{vmatrix}}{\begin{vmatrix} G_1 & G_{-1} & G_{-3} \\ G_3 & G_1 & G_{-1} \\ G_5 - G_1^2G_3 & G_3 & G_1 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2$$

Example 3.7.6 (Chen Shuwen, 2020)

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k \quad (k = -3, -1, 1, 3, 5, 9)$$

and

$$G_{-1} = P_{-1}$$

$$G_{-3} = -(P_{-1}^3 - P_{-3})/3$$

$$G_1 = -P_1$$

$$G_3 = (P_1^3 - P_3)/3$$

$$G_5 = (P_1^5 - P_5)/5$$

$$G_9 = (P_1^9 - P_9)/9 - (G_3^3/3)$$

then

$$G_1^4 + 4 \frac{\begin{vmatrix} G_3(1 + G_{-1}G_1) & G_{-1} & G_{-3} \\ G_5 & G_1 & G_{-1} \\ G_9 - G_1^4G_5 - G_1G_3G_5 & G_5 & G_1^3 + G_3 \end{vmatrix}}{\begin{vmatrix} G_1 & G_{-1} & G_{-3} \\ G_3 & G_1 & G_{-1} \\ G_5 - G_1^2G_3 & G_3 & G_1 \end{vmatrix}} = (a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2)^2$$

Example 3.7.7 (Chen Shuwen, 2020)

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k \quad (k = -5, -3, -1, 1, 3, 5)$$

and

$$G_{-1} = P_{-1}$$

$$G_{-3} = -(P_{-1}^3 - P_{-3})/3$$

$$G_{-5} = -(P_{-1}^5 - P_{-5})/5$$

$$G_1 = -P_1$$

$$G_3 = (P_1^3 - P_3)/3$$

$$G_5 = (P_1^5 - P_5)/5$$

then

$$G_1^2 + 2 \frac{\begin{vmatrix} G_1 & G_{-3} & G_{-5} - G_{-1}^2G_{-3} \\ G_3 & G_{-1} & G_{-3} \\ G_5 - G_1^2G_3 & G_1 & G_{-1} \end{vmatrix}}{\begin{vmatrix} G_{-1} & G_{-3} & G_{-5} - G_{-1}^2G_{-3} \\ G_1 & G_{-1} & G_{-3} \\ G_3 & G_1 & G_{-1} \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2$$

Example 3.7.8 (Chen Shuwen, 2020)

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k \quad (k = -5, -3, -1, 1, 3, 7)$$

and

$$\begin{aligned} G_{-1} &= P_{-1} \\ G_{-3} &= -(P_{-1}^3 - P_{-3})/3 \\ G_{-5} &= -(P_{-1}^5 - P_{-5})/5 \\ G_1 &= -P_1 \\ G_3 &= (P_1^3 - P_3)/3 \\ G_7 &= (P_1^7 - P_7)/7 \end{aligned}$$

then

$$G_1^4 + 4 \frac{\begin{vmatrix} G_1 & G_{-3} & G_{-5} - G_{-1}^2 G_{-3} \\ G_3 & G_{-1} & G_{-3} \\ G_7 - G_1^4 G_3 - G_1 G_3^2 & G_1^3 + G_3 & G_1(1 + G_{-1} G_1) \end{vmatrix}}{\begin{vmatrix} G_{-1} & G_{-3} & G_{-5} - G_{-1}^2 G_{-3} \\ G_1 & G_{-1} & G_{-3} \\ G_3 & G_1 & G_{-1} \end{vmatrix}} = (a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2)^2$$

Example 3.8.1 (Chen Shuwen, 2020)

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k + a_7^k \quad (k = 1, 3, 5, 7, 9, 11, 13)$$

and

$$\begin{aligned} G_1 &= -P_1 \\ G_3 &= (P_1^3 - P_3)/3 \\ G_5 &= (P_1^5 - P_5)/5 \\ G_7 &= (P_1^7 - P_7)/7 \\ G_9 &= (P_1^9 - P_9)/9 - (G_3^3/3) \\ G_{11} &= (P_1^{11} - P_{11})/11 - (G_3^2 G_5) \\ G_{13} &= (P_1^{13} - P_{13})/13 - (G_3 G_5^2 + G_3^2 G_7) \end{aligned}$$

then

$$G_1^2 + 2 \frac{\begin{vmatrix} G_7 & G_3 & 0 & 1 \\ G_9 & G_5 & G_3 & G_1^2 \\ G_{11} & G_7 & G_5 & G_1^4 + G_3 G_1 \\ G_{13} & G_9 & G_7 & G_1^6 + G_3 G_1^3 + G_5 G_1 \end{vmatrix}}{\begin{vmatrix} G_5 & G_3 & 0 & 1 \\ G_7 & G_5 & G_3 & G_1^2 \\ G_9 & G_7 & G_5 & G_1^4 + G_3 G_1 \\ G_{11} & G_9 & G_7 & G_1^6 + G_3 G_1^3 + G_5 G_1 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2$$

Example 3.8.2 (Chen Shuwen, 2020)

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k + a_7^k + a_8^k \quad (k = 1, 3, 5, 7, 9, 11, 13, 15)$$

and

$$\begin{aligned} G_1 &= -P_1 \\ G_3 &= (P_1^3 - P_3)/3 \end{aligned}$$

$$\begin{aligned}
 G_5 &= (P_1^5 - P_5)/5 \\
 G_7 &= (P_1^7 - P_7)/7 \\
 G_9 &= (P_1^9 - P_9)/9 - (G_3^3/3) \\
 G_{11} &= (P_1^{11} - P_{11})/11 - (G_3^2 G_5) \\
 G_{13} &= (P_1^{13} - P_{13})/13 - (G_3 G_5^2 + G_3^2 G_7) \\
 G_{15} &= (P_1^{15} - P_{15})/15 - (G_3^5/5 + G_5^3/3 + 2G_3 G_5 G_7 + G_3^2 G_9)
 \end{aligned}$$

then

$$G_1^2 + 2 \frac{\begin{vmatrix} G_9 & G_5 & G_3 & & G_1 \\ G_{11} & G_7 & G_5 & & G_1^3 + G_3 \\ G_{13} & G_9 & G_7 & & G_1^5 + G_3 G_1^2 + G_5 \\ G_{15} & G_{11} & G_9 & & G_1^7 + G_3 G_1^4 + G_5 G_1^2 + G_7 \\ G_7 & G_5 & G_3 & & G_1 \\ G_9 & G_7 & G_5 & & G_1^3 + G_3 \\ G_{11} & G_9 & G_7 & & G_1^5 + G_3 G_1^2 + G_5 \\ G_{13} & G_{11} & G_9 & & G_1^7 + G_3 G_1^4 + G_5 G_1^2 + G_7 \end{vmatrix}}{\begin{vmatrix} G_7 & G_5 & G_3 & & G_1 \\ G_9 & G_7 & G_5 & & G_1^3 + G_3 \\ G_{11} & G_9 & G_7 & & G_1^5 + G_3 G_1^2 + G_5 \\ G_{13} & G_{11} & G_9 & & G_1^7 + G_3 G_1^4 + G_5 G_1^2 + G_7 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2$$

Example 3.8.3 (Chen Shuwen, 2020)

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k + a_7^k + a_8^k + a_9^k \quad (k = 1, 3, 5, 7, 9, 11, 13, 15, 17)$$

and

$$\begin{aligned}
 G_1 &= -P_1 \\
 G_3 &= (P_1^3 - P_3)/3 \\
 G_5 &= (P_1^5 - P_5)/5 \\
 G_7 &= (P_1^7 - P_7)/7 \\
 G_9 &= (P_1^9 - P_9)/9 - (G_3^3/3) \\
 G_{11} &= (P_1^{11} - P_{11})/11 - (G_3^2 G_5) \\
 G_{13} &= (P_1^{13} - P_{13})/13 - (G_3 G_5^2 + G_3^2 G_7) \\
 G_{15} &= (P_1^{15} - P_{15})/15 - (G_3^5/5 + G_5^3/3 + 2G_3 G_5 G_7 + G_3^2 G_9) \\
 G_{17} &= (P_1^{17} - P_{17})/17 - (G_3^4 G_5 + G_5^2 G_7 + G_3 G_7^2 + 2G_3 G_5 G_9 + G_3^2 G_{11})
 \end{aligned}$$

then

$$G_1^2 + 2 \frac{\begin{vmatrix} G_9 & G_5 & G_3 & 0 & & 1 \\ G_{11} & G_7 & G_5 & G_3 & & G_1^2 \\ G_{13} & G_9 & G_7 & G_5 & & G_1^4 + G_3 G_1 \\ G_{15} & G_{11} & G_9 & G_7 & & G_1^6 + G_3 G_1^3 + G_5 G_1 \\ G_{17} & G_{13} & G_{11} & G_9 & & G_1^8 + G_3 G_1^5 + G_5 G_1^3 + G_7 G_1 \\ G_7 & G_5 & G_3 & 0 & & 1 \\ G_9 & G_7 & G_5 & G_3 & & G_1^2 \\ G_{11} & G_9 & G_7 & G_5 & & G_1^4 + G_3 G_1 \\ G_{13} & G_{11} & G_9 & G_7 & & G_1^6 + G_3 G_1^3 + G_5 G_1 \\ G_{15} & G_{13} & G_{11} & G_9 & & G_1^8 + G_3 G_1^5 + G_5 G_1^3 + G_7 G_1 \end{vmatrix}}{\begin{vmatrix} G_7 & G_5 & G_3 & 0 & & 1 \\ G_9 & G_7 & G_5 & G_3 & & G_1^2 \\ G_{11} & G_9 & G_7 & G_5 & & G_1^4 + G_3 G_1 \\ G_{13} & G_{11} & G_9 & G_7 & & G_1^6 + G_3 G_1^3 + G_5 G_1 \\ G_{15} & G_{13} & G_{11} & G_9 & & G_1^8 + G_3 G_1^5 + G_5 G_1^3 + G_7 G_1 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + \dots + a_9^2$$

Example 3.8.4 (Chen Shuwen, 2020)

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k + a_7^k + a_8^k + a_9^k + a_{10}^k \quad (k = 1, 3, 5, 7, 9, 11, 13, 15, 17, 19)$$

and

$$G_1 = -P_1$$

$$G_3 = (P_1^3 - P_3)/3$$

$$G_5 = (P_1^5 - P_5)/5$$

$$G_7 = (P_1^7 - P_7)/7$$

$$G_9 = (P_1^9 - P_9)/9 - (G_3^3/3)$$

$$G_{11} = (P_1^{11} - P_{11})/11 - (G_3^2 G_5)$$

$$G_{13} = (P_1^{13} - P_{13})/13 - (G_3 G_5^2 + G_3^2 G_7)$$

$$G_{15} = (P_1^{15} - P_{15})/15 - (G_3^5/5 + G_5^3/3 + 2G_3 G_5 G_7 + G_3^2 G_9)$$

$$G_{17} = (P_1^{17} - P_{17})/17 - (G_3^4 G_5 + G_5^2 G_7 + G_3 G_7^2 + 2G_3 G_5 G_9 + G_3^2 G_{11})$$

$$G_{19} = (P_1^{19} - P_{19})/19 - (2G_3^3 G_5^2 + G_3^4 G_7 + G_5 G_7^2 + G_5^2 G_9 + 2G_3 G_7 G_9 + 2G_3 G_5 G_{11} + G_3^2 G_{13})$$

then

$$G_1^2 + 2 \frac{\begin{vmatrix} G_{11} & G_7 & G_5 & G_3 & G_1 \\ G_{13} & G_9 & G_7 & G_5 & G_1^3 + G_3 \\ G_{15} & G_{11} & G_9 & G_7 & G_1^5 + G_3 G_1^2 + G_5 \\ G_{17} & G_{13} & G_{11} & G_9 & G_1^7 + G_3 G_1^4 + G_5 G_1^2 + G_7 \\ G_{19} & G_{15} & G_{13} & G_{11} & G_1^9 + G_3 G_1^6 + G_5 G_1^4 + G_7 G_1^2 + G_9 \end{vmatrix}}{\begin{vmatrix} G_9 & G_7 & G_5 & G_3 & G_1 \\ G_{11} & G_9 & G_7 & G_5 & G_1^3 + G_3 \\ G_{13} & G_{11} & G_9 & G_7 & G_1^5 + G_3 G_1^2 + G_5 \\ G_{15} & G_{13} & G_{11} & G_9 & G_1^7 + G_3 G_1^4 + G_5 G_1^2 + G_7 \\ G_{17} & G_{15} & G_{13} & G_{11} & G_1^9 + G_3 G_1^6 + G_5 G_1^4 + G_7 G_1^2 + G_9 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + \dots + a_{10}^2$$

Identities 3.12. (The generalization of the Girard-Newton Identities, Chen Shuwen, 1997, 2017, 2020)

Let n and m be positive integers, k be integer. Denote

$$R_k = \begin{cases} a_1^k + a_2^k + \cdots + a_n^k - (b_1^k + b_2^k + \cdots + b_m^k) & \text{for } k \neq 0 \\ -\frac{b_1 b_2 \cdots b_m}{a_1 a_2 \cdots a_n} & \text{for } k = 0 \end{cases}$$

and let

$$\begin{cases} U_1 = R_1 \\ U_k = (R_k + U_1 R_{k-1} + U_2 R_{k-2} + \cdots + U_{k-1} R_1)/k, & \text{for } k > 1 \\ U_0 = R_0 \\ U_{-1} = R_0 R_{-1} \\ U_k = (R_0 R_k + U_{-1} R_{k+1} + U_{-2} R_{k+2} + \cdots + U_{k+1} R_{-1})/(-k), & \text{for } k < -1 \end{cases}$$

and let

$$S_k = \gamma_k + \delta_k \quad \text{where } \gamma_k = \begin{cases} 0 & (k < 0) \\ 1 & (k = 0) \\ U_k & (k > 0) \end{cases}, \quad \delta_k = \begin{cases} (-1)^{m-n} U_{k-m+n} & (k \leq m-n) \\ 0 & (k > m-n) \end{cases}$$

and for $0 \leq t < n$, let

$$W_{nk+t} = \begin{vmatrix} D_k \\ D_{k+1} \\ D_{k+2} \\ \vdots \\ D_{k+n-t-1} \\ D_{k+n-t+1} \\ \vdots \\ D_{k+n} \\ D_{k+n+1} \end{vmatrix}, \text{ where } D_k = [S_k \quad S_{k-1} \quad S_{k-2} \quad \cdots \quad S_{k-n+2} \quad S_{k-n+1}]$$

Then

$$W_{nm} = \prod_{i=1}^n \prod_{j=1}^m (a_i - b_j)$$

and

$$\begin{cases} \frac{W_{nk+1}}{W_{nk}} = a_1 + a_2 + \cdots + a_n \\ \frac{W_{nk+2}}{W_{nk}} = a_1 a_2 + a_1 a_3 + \cdots + a_{n-1} a_n \\ \frac{W_{nk+3}}{W_{nk}} = a_1 a_2 a_3 + a_1 a_2 a_4 + \cdots + a_{n-2} a_{n-1} a_n \\ \vdots \\ \frac{W_{nk+n}}{W_{nk}} = a_1 a_2 a_3 \cdots a_n \end{cases}$$

Remark. Here we generalize The Girard-Newton Identities from the case of $k > 0$ and $m = 0$ to the case of any integer k and any positive integer m, n . It is useful for solving the Generalization of Prouhet-Tarry-Escott Problem and Equal sums of like powers system, which will be discussed in Chapter 5.

Example 3.9.1 (Chen Shuwen, 2017, 2019, 2020)

Let

$$\left\{ \begin{array}{l} R_1 = a_1 + a_2 + a_3 + a_4 - b_1 - b_2 - b_3 - b_4 - b_5 - b_6 \\ R_2 = a_1^2 + a_2^2 + a_3^2 + a_4^2 - b_1^2 - b_2^2 - b_3^2 - b_4^2 - b_5^2 - b_6^2 \\ R_3 = a_1^3 + a_2^3 + a_3^3 + a_4^3 - b_1^3 - b_2^3 - b_3^3 - b_4^3 - b_5^3 - b_6^3 \\ R_4 = a_1^4 + a_2^4 + a_3^4 + a_4^4 - b_1^4 - b_2^4 - b_3^4 - b_4^4 - b_5^4 - b_6^4 \\ R_5 = a_1^5 + a_2^5 + a_3^5 + a_4^5 - b_1^5 - b_2^5 - b_3^5 - b_4^5 - b_5^5 - b_6^5 \\ R_6 = a_1^6 + a_2^6 + a_3^6 + a_4^6 - b_1^6 - b_2^6 - b_3^6 - b_4^6 - b_5^6 - b_6^6 \\ R_7 = a_1^7 + a_2^7 + a_3^7 + a_4^7 - b_1^7 - b_2^7 - b_3^7 - b_4^7 - b_5^7 - b_6^7 \\ R_8 = a_1^8 + a_2^8 + a_3^8 + a_4^8 - b_1^8 - b_2^8 - b_3^8 - b_4^8 - b_5^8 - b_6^8 \\ R_9 = a_1^9 + a_2^9 + a_3^9 + a_4^9 - b_1^9 - b_2^9 - b_3^9 - b_4^9 - b_5^9 - b_6^9 \\ R_{10} = a_1^{10} + a_2^{10} + a_3^{10} + a_4^{10} - b_1^{10} - b_2^{10} - b_3^{10} - b_4^{10} - b_5^{10} - b_6^{10} \end{array} \right.$$

And

$$\left\{ \begin{array}{l} S_1 = (R_1)/1 \\ S_2 = (R_2 + S_1R_1)/2 \\ S_3 = (R_3 + S_1R_2 + S_2R_1)/3 \\ S_4 = (R_4 + S_1R_3 + S_2R_2 + S_3R_1)/4 \\ S_5 = (R_5 + S_1R_4 + S_2R_3 + S_3R_2 + S_4R_1)/5 \\ S_6 = (R_6 + S_1R_5 + S_2R_4 + S_3R_3 + S_4R_2 + S_5R_1)/6 \\ S_7 = (R_7 + S_1R_6 + S_2R_5 + S_3R_4 + S_4R_3 + S_5R_2 + S_6R_1)/7 \\ S_8 = (R_8 + S_1R_7 + S_2R_6 + S_3R_5 + S_4R_4 + S_5R_3 + S_6R_2 + S_7R_1)/8 \\ S_9 = (R_9 + S_1R_8 + S_2R_7 + S_3R_6 + S_4R_5 + S_5R_4 + S_6R_3 + S_7R_2 + S_8R_1)/9 \\ S_{10} = (R_{10} + S_1R_9 + S_2R_8 + S_3R_7 + S_4R_6 + S_5R_5 + S_6R_4 + S_7R_3 + S_8R_2 + S_9R_1)/10 \end{array} \right.$$

and

$$\begin{array}{l} W_{24} = \begin{vmatrix} S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \\ S_8 & S_7 & S_6 & S_5 \\ S_9 & S_8 & S_7 & S_6 \end{vmatrix} \\ W_{25} = \begin{vmatrix} S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \\ S_8 & S_7 & S_6 & S_5 \\ S_{10} & S_9 & S_8 & S_7 \end{vmatrix} \\ W_{26} = \begin{vmatrix} S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \\ S_9 & S_8 & S_7 & S_6 \\ S_{10} & S_9 & S_8 & S_7 \end{vmatrix} \\ W_{27} = \begin{vmatrix} S_6 & S_5 & S_4 & S_3 \\ S_8 & S_7 & S_6 & S_5 \\ S_9 & S_8 & S_7 & S_6 \\ S_{10} & S_9 & S_8 & S_7 \end{vmatrix} \\ W_{28} = \begin{vmatrix} S_7 & S_6 & S_5 & S_4 \\ S_8 & S_7 & S_6 & S_5 \\ S_9 & S_8 & S_7 & S_6 \\ S_{10} & S_9 & S_8 & S_7 \end{vmatrix} \end{array}$$

Then

$$\begin{array}{ll} \frac{W_{25}}{W_{24}} = a_1 + a_2 + a_3 + a_4 & \frac{W_{26}}{W_{24}} = a_1a_2 + a_1a_3 + a_2a_3 + a_1a_4 + a_2a_4 + a_3a_4 \\ \frac{W_{27}}{W_{24}} = a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4 & \frac{W_{28}}{W_{24}} = a_1a_2a_3a_4 \end{array}$$

Remark. This identities is for solving the Prouhet-Tarry-Escott Problem of the below case.

$$\begin{aligned} & a_0^k + a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k + a_7^k + a_8^k + a_9^k + a_{10}^k \\ & = b_0^k + b_1^k + b_2^k + b_3^k + b_4^k + b_5^k + b_6^k + b_7^k + b_8^k + b_9^k + b_{10}^k \\ & (k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10) \end{aligned}$$

Remark. For the above Example 3.9

$$\begin{aligned}
 S_1 &= a_1 + a_2 + a_3 + a_4 - b_1 - b_2 - b_3 - b_4 - b_5 - b_6 \\
 S_2 &= a_1^2 + a_1a_2 + a_2^2 + a_1a_3 + a_2a_3 + a_3^2 + a_1a_4 + a_2a_4 + a_3a_4 + a_4^2 - a_1b_1 - a_2b_1 - a_3b_1 - a_4b_1 - a_1b_2 \\
 &\quad - a_2b_2 - a_3b_2 - a_4b_2 + b_1b_2 - a_1b_3 - a_2b_3 - a_3b_3 - a_4b_3 + b_1b_3 + b_2b_3 - a_1b_4 - a_2b_4 \\
 &\quad - a_3b_4 - a_4b_4 + b_1b_4 + b_2b_4 + b_3b_4 - a_1b_5 - a_2b_5 - a_3b_5 - a_4b_5 + b_1b_5 + b_2b_5 + b_3b_5 \\
 &\quad + b_4b_5 - a_1b_6 - a_2b_6 - a_3b_6 - a_4b_6 + b_1b_6 + b_2b_6 + b_3b_6 + b_4b_6 + b_5b_6 \\
 S_3 &= a_1^3 + a_1^2a_2 + a_1a_2^2 + a_2^3 + a_1^2a_3 + a_1a_2a_3 + a_2^2a_3 + a_1a_3^2 + a_2a_3^2 + a_3^3 + a_1^2a_4 + a_1a_2a_4 + a_2^2a_4 + a_1a_3a_4 \\
 &\quad + \dots + a_3b_5b_6 + a_4b_5b_6 - b_1b_5b_6 - b_2b_5b_6 - b_3b_5b_6 - b_4b_5b_6
 \end{aligned}$$

$$\dots \dots$$

$$S_k = \sum_{\substack{0 \leq k_1, k_2, k_3, k_4 \leq k \\ 0 \leq k_5, k_6, k_7, k_8, k_9, k_{10} \leq 1 \\ k_1 + k_2 + \dots + k_9 + k_{10} = k}} a_1^{k_1} a_2^{k_2} a_3^{k_3} a_4^{k_4} (-b_1)^{k_5} (-b_2)^{k_6} (-b_3)^{k_7} (-b_4)^{k_8} (-b_5)^{k_9} (-b_6)^{k_{10}}$$

and

$$\begin{aligned}
 W_{24} &= S_6^4 - 3S_5S_6^2S_7 + S_5^2S_7^2 + 2S_4S_6S_7^2 - S_3S_7^3 + 2S_5^2S_6S_8 - 2S_4S_6^2S_8 - 2S_4S_5S_7S_8 + 2S_3S_6S_7S_8 + S_4^2S_8^2 \\
 &\quad - S_3S_5S_8^2 - S_5^3S_9 + 2S_4S_5S_6S_9 - S_3S_6^2S_9 - S_4^2S_7S_9 + S_3S_5S_7S_9 \\
 W_{25} &= S_6^3S_7 - 2S_5S_6S_7^2 + S_4S_7^3 - S_5S_6^2S_8 + 2S_5^2S_7S_8 - S_3S_7^2S_8 - S_4S_5S_8^2 + S_3S_6S_8^2 + S_5^2S_6S_9 - S_4S_6^2S_9 \\
 &\quad - S_4S_5S_7S_9 + S_3S_6S_7S_9 + S_4^2S_8S_9 - S_3S_5S_8S_9 - S_5^3S_{10} + 2S_4S_5S_6S_{10} - S_3S_6^2S_{10} - S_4^2S_7S_{10} \\
 &\quad + S_3S_5S_7S_{10} \\
 W_{26} &= S_6^2S_7^2 - S_5S_7^3 - S_6^3S_8 + S_4S_7^2S_8 + S_4S_6S_8^2 - S_3S_7S_8^2 + S_5S_6^2S_9 + S_5^2S_7S_9 - 3S_4S_6S_7S_9 + S_3S_7^2S_9 - S_4S_5S_8S_9 \\
 &\quad + S_3S_6S_8S_9 + S_4^2S_9^2 - S_3S_5S_9^2 - S_5^2S_6S_{10} + S_4S_6^2S_{10} + S_4S_5S_7S_{10} - S_3S_6S_7S_{10} - S_4^2S_8S_{10} \\
 &\quad + S_3S_5S_8S_{10} \\
 W_{27} &= S_6S_7^3 - 2S_6^2S_7S_8 - S_5S_7^2S_8 + 2S_5S_6S_8^2 + S_4S_7S_8^2 - S_3S_8^3 + S_6^3S_9 - S_4S_7^2S_9 - S_5^2S_8S_9 - S_4S_6S_8S_9 \\
 &\quad + 2S_3S_7S_8S_9 + S_4S_5S_9^2 - S_3S_6S_9^2 - S_5S_6^2S_{10} + S_5^2S_7S_{10} + S_4S_6S_7S_{10} - S_3S_7^2S_{10} - S_4S_5S_8S_{10} \\
 &\quad + S_3S_6S_8S_{10} \\
 W_{28} &= S_7^4 - 3S_6S_7^2S_8 + S_6^2S_8^2 + 2S_5S_7S_8^2 - S_4S_8^3 + 2S_6^2S_7S_9 - 2S_5S_7^2S_9 - 2S_5S_6S_8S_9 + 2S_4S_7S_8S_9 + S_5^2S_9^2 \\
 &\quad - S_4S_6S_9^2 - S_6^3S_{10} + 2S_5S_6S_7S_{10} - S_4S_7^2S_{10} - S_5^2S_8S_{10} + S_4S_6S_8S_{10}
 \end{aligned}$$

and

$$\begin{aligned}
 W_{24} &= (a_1 - b_1)(a_1 - b_2)(a_1 - b_3)(a_1 - b_4)(a_1 - b_5)(a_1 - b_6) \\
 &\quad \times (a_2 - b_1)(a_2 - b_2)(a_2 - b_3)(a_2 - b_4)(a_2 - b_5)(a_2 - b_6) \\
 &\quad \times (a_3 - b_1)(a_3 - b_2)(a_3 - b_3)(a_3 - b_4)(a_3 - b_5)(a_3 - b_6) \\
 &\quad \times (a_4 - b_1)(a_4 - b_2)(a_4 - b_3)(a_4 - b_4)(a_4 - b_5)(a_4 - b_6)
 \end{aligned}$$

Example 3.9.2 (Chen Shuwen, 2017, 2019, 2020)

Let

$$\left\{ \begin{aligned}
 R_1 &= a_1 + a_2 + a_3 + a_4 - b_1 - b_2 - b_3 - b_4 \\
 R_2 &= a_1^2 + a_2^2 + a_3^2 + a_4^2 - b_1^2 - b_2^2 - b_3^2 - b_4^2 \\
 R_3 &= a_1^3 + a_2^3 + a_3^3 + a_4^3 - b_1^3 - b_2^3 - b_3^3 - b_4^3 \\
 R_4 &= a_1^4 + a_2^4 + a_3^4 + a_4^4 - b_1^4 - b_2^4 - b_3^4 - b_4^4 \\
 R_5 &= a_1^5 + a_2^5 + a_3^5 + a_4^5 - b_1^5 - b_2^5 - b_3^5 - b_4^5 \\
 R_6 &= a_1^6 + a_2^6 + a_3^6 + a_4^6 - b_1^6 - b_2^6 - b_3^6 - b_4^6 \\
 R_7 &= a_1^7 + a_2^7 + a_3^7 + a_4^7 - b_1^7 - b_2^7 - b_3^7 - b_4^7 \\
 R_8 &= a_1^8 + a_2^8 + a_3^8 + a_4^8 - b_1^8 - b_2^8 - b_3^8 - b_4^8
 \end{aligned} \right.$$

and

$$\left\{ \begin{array}{l} S_1 = (R_1)/1 \\ S_2 = (R_2 + S_1R_1)/2 \\ S_3 = (R_3 + S_1R_2 + S_2R_1)/3 \\ S_4 = (R_4 + S_1R_3 + S_2R_2 + S_3R_1)/4 \\ S_5 = (R_5 + S_1R_4 + S_2R_3 + S_3R_2 + S_4R_1)/5 \\ S_6 = (R_6 + S_1R_5 + S_2R_4 + S_3R_3 + S_4R_2 + S_5R_1)/6 \\ S_7 = (R_7 + S_1R_6 + S_2R_5 + S_3R_4 + S_4R_3 + S_5R_2 + S_6R_1)/7 \\ S_8 = (R_8 + S_1R_7 + S_2R_6 + S_3R_5 + S_4R_4 + S_5R_3 + S_6R_2 + S_7R_1)/8 \end{array} \right.$$

and

$$W_{16} = \begin{vmatrix} S_4 & S_3 & S_2 & S_1 \\ S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \end{vmatrix} \quad W_{17} = \begin{vmatrix} S_4 & S_3 & S_2 & S_1 \\ S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \\ S_8 & S_7 & S_6 & S_5 \end{vmatrix} \quad W_{18} = \begin{vmatrix} S_4 & S_3 & S_2 & S_1 \\ S_5 & S_4 & S_3 & S_2 \\ S_7 & S_6 & S_5 & S_4 \\ S_8 & S_7 & S_6 & S_5 \end{vmatrix}$$

$$W_{19} = \begin{vmatrix} S_4 & S_3 & S_2 & S_1 \\ S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \\ S_8 & S_7 & S_6 & S_5 \end{vmatrix} \quad W_{20} = \begin{vmatrix} S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \\ S_8 & S_7 & S_6 & S_5 \end{vmatrix}$$

Then

$$\begin{aligned} W_{16} &= (a_1 - b_1)(a_1 - b_2)(a_1 - b_3)(a_1 - b_4) \\ &\quad \times (a_2 - b_1)(a_2 - b_2)(a_2 - b_3)(a_2 - b_4) \\ &\quad \times (a_3 - b_1)(a_3 - b_2)(a_3 - b_3)(a_3 - b_4) \\ &\quad \times (a_4 - b_1)(a_4 - b_2)(a_4 - b_3)(a_4 - b_4) \end{aligned}$$

and

$$\frac{W_{17}}{W_{16}} = a_1 + a_2 + a_3 + a_4 \quad \frac{W_{18}}{W_{16}} = a_1a_2 + a_1a_3 + a_2a_3 + a_1a_4 + a_2a_4 + a_3a_4$$

$$\frac{W_{19}}{W_{16}} = a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4 \quad \frac{W_{20}}{W_{16}} = a_1a_2a_3a_4$$

Example 3.9.3 (Chen Shuwen, 2017, 2019, 2020)

Let

$$\left\{ \begin{array}{l} R_1 = a_1 + a_2 + a_3 + a_4 - b_1 - b_2 - b_3 - b_4 \\ R_2 = a_1^2 + a_2^2 + a_3^2 + a_4^2 - b_1^2 - b_2^2 - b_3^2 - b_4^2 \\ R_3 = a_1^3 + a_2^3 + a_3^3 + a_4^3 - b_1^3 - b_2^3 - b_3^3 - b_4^3 \\ R_4 = a_1^4 + a_2^4 + a_3^4 + a_4^4 - b_1^4 - b_2^4 - b_3^4 - b_4^4 \\ R_5 = a_1^5 + a_2^5 + a_3^5 + a_4^5 - b_1^5 - b_2^5 - b_3^5 - b_4^5 \\ R_6 = a_1^6 + a_2^6 + a_3^6 + a_4^6 - b_1^6 - b_2^6 - b_3^6 - b_4^6 \\ R_7 = a_1^7 + a_2^7 + a_3^7 + a_4^7 - b_1^7 - b_2^7 - b_3^7 - b_4^7 \\ R_8 = a_1^8 + a_2^8 + a_3^8 + a_4^8 - b_1^8 - b_2^8 - b_3^8 - b_4^8 \\ R_9 = a_1^9 + a_2^9 + a_3^9 + a_4^9 - b_1^9 - b_2^9 - b_3^9 - b_4^9 \\ R_0 = -\frac{b_1 b_2 b_3 b_4}{a_1 a_2 a_3 a_4} \\ R_{-1} = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} - \frac{1}{b_1} - \frac{1}{b_2} - \frac{1}{b_3} - \frac{1}{b_4} \\ R_{-2} = \frac{1}{a_1^2} + \frac{1}{a_2^2} + \frac{1}{a_3^2} + \frac{1}{a_4^2} - \frac{1}{b_1^2} - \frac{1}{b_2^2} - \frac{1}{b_3^2} - \frac{1}{b_4^2} \\ R_{-3} = \frac{1}{a_1^3} + \frac{1}{a_2^3} + \frac{1}{a_3^3} + \frac{1}{a_4^3} - \frac{1}{b_1^3} - \frac{1}{b_2^3} - \frac{1}{b_3^3} - \frac{1}{b_4^3} \end{array} \right.$$

and

$$\left\{ \begin{array}{l} S_1 = (R_1)/1 \\ S_2 = (R_2 + S_1 R_1)/2 \\ S_3 = (R_3 + S_1 R_2 + S_2 R_1)/3 \\ S_4 = (R_4 + S_1 R_3 + S_2 R_2 + S_3 R_1)/4 \\ S_5 = (R_5 + S_1 R_4 + S_2 R_3 + S_3 R_2 + S_4 R_1)/5 \\ S_6 = (R_6 + S_1 R_5 + S_2 R_4 + S_3 R_3 + S_4 R_2 + S_5 R_1)/6 \\ S_7 = (R_7 + S_1 R_6 + S_2 R_5 + S_3 R_4 + S_4 R_3 + S_5 R_2 + S_6 R_1)/7 \\ S_8 = (R_8 + S_1 R_7 + S_2 R_6 + S_3 R_5 + S_4 R_4 + S_5 R_3 + S_6 R_2 + S_7 R_1)/8 \\ S_9 = (R_9 + S_1 R_8 + S_2 R_7 + S_3 R_6 + S_4 R_5 + S_5 R_4 + S_6 R_3 + S_7 R_2 + S_8 R_1)/9 \\ S_0 = 1 + R_0 \\ S_{-1} = (R_0 R_{-1})/1 \\ S_{-2} = (R_0 R_{-2} + S_{-1} R_{-1})/2 \\ S_{-3} = (R_0 R_{-3} + S_{-1} R_{-2} + S_{-2} R_{-1})/3 \end{array} \right.$$

and

$$\begin{aligned} W_8 &= \begin{vmatrix} S_2 & S_1 & S_0 & S_{-1} \\ S_3 & S_2 & S_1 & S_0 \\ S_4 & S_3 & S_2 & S_1 \\ S_5 & S_4 & S_3 & S_2 \end{vmatrix} & W_9 &= \begin{vmatrix} S_2 & S_1 & S_0 & S_{-1} \\ S_3 & S_2 & S_1 & S_0 \\ S_4 & S_3 & S_2 & S_1 \\ S_6 & S_5 & S_4 & S_3 \end{vmatrix} & W_{10} &= \begin{vmatrix} S_2 & S_1 & S_0 & S_{-1} \\ S_3 & S_2 & S_1 & S_0 \\ S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \end{vmatrix} & W_{11} &= \begin{vmatrix} S_2 & S_1 & S_0 & S_{-1} \\ S_4 & S_3 & S_2 & S_1 \\ S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \end{vmatrix} \\ \\ W_{12} &= \begin{vmatrix} S_3 & S_2 & S_1 & S_0 \\ S_4 & S_3 & S_2 & S_1 \\ S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \end{vmatrix} & W_{13} &= \begin{vmatrix} S_3 & S_2 & S_1 & S_0 \\ S_4 & S_3 & S_2 & S_1 \\ S_5 & S_4 & S_3 & S_2 \\ S_7 & S_6 & S_5 & S_4 \end{vmatrix} & W_{14} &= \begin{vmatrix} S_3 & S_2 & S_1 & S_0 \\ S_4 & S_3 & S_2 & S_1 \\ S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \end{vmatrix} & W_{15} &= \begin{vmatrix} S_3 & S_2 & S_1 & S_0 \\ S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \end{vmatrix} \end{aligned}$$

$$\begin{aligned}
W_{16} &= \begin{vmatrix} S_4 & S_3 & S_2 & S_1 \\ S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \end{vmatrix} & W_{17} &= \begin{vmatrix} S_4 & S_3 & S_2 & S_1 \\ S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \\ S_8 & S_7 & S_6 & S_5 \end{vmatrix} & W_{18} &= \begin{vmatrix} S_4 & S_3 & S_2 & S_1 \\ S_5 & S_4 & S_3 & S_2 \\ S_7 & S_6 & S_5 & S_4 \\ S_8 & S_7 & S_6 & S_5 \end{vmatrix} & W_{19} &= \begin{vmatrix} S_4 & S_3 & S_2 & S_1 \\ S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \\ S_8 & S_7 & S_6 & S_5 \end{vmatrix} \\
W_{20} &= \begin{vmatrix} S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \\ S_8 & S_7 & S_6 & S_5 \end{vmatrix} & W_{21} &= \begin{vmatrix} S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \\ S_9 & S_8 & S_7 & S_6 \end{vmatrix} & W_{22} &= \begin{vmatrix} S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \\ S_8 & S_7 & S_6 & S_5 \\ S_9 & S_8 & S_7 & S_6 \end{vmatrix} & W_{23} &= \begin{vmatrix} S_5 & S_4 & S_3 & S_2 \\ S_7 & S_6 & S_5 & S_4 \\ S_8 & S_7 & S_6 & S_5 \\ S_9 & S_8 & S_7 & S_6 \end{vmatrix}
\end{aligned}$$

Then

$$W_8/W_{16} = 1/(a_1^2 a_2^2 a_3^2 a_4^2)$$

$$W_9/W_{16} = (a_1 + a_2 + a_3 + a_4)/(a_1^2 a_2^2 a_3^2 a_4^2)$$

$$W_{10}/W_{16} = (a_1 a_2 + a_1 a_3 + a_2 a_3 + a_1 a_4 + a_2 a_4 + a_3 a_4)/(a_1^2 a_2^2 a_3^2 a_4^2)$$

$$W_{11}/W_{16} = (a_1 a_2 a_3 + a_1 a_2 a_4 + a_1 a_3 a_4 + a_2 a_3 a_4)/(a_1^2 a_2^2 a_3^2 a_4^2)$$

$$W_{12}/W_{16} = 1/(a_1 a_2 a_3 a_4)$$

$$W_{13}/W_{16} = (a_1 + a_2 + a_3 + a_4)/(a_1 a_2 a_3 a_4)$$

$$W_{14}/W_{16} = (a_1 a_2 + a_1 a_3 + a_2 a_3 + a_1 a_4 + a_2 a_4 + a_3 a_4)/(a_1 a_2 a_3 a_4)$$

$$W_{15}/W_{16} = (a_1 a_2 a_3 + a_1 a_2 a_4 + a_1 a_3 a_4 + a_2 a_3 a_4)/(a_1 a_2 a_3 a_4)$$

$$W_{16} = (a_1 - b_1)(a_2 - b_1)(a_3 - b_1)(a_4 - b_1)(a_1 - b_2)(a_2 - b_2)(a_3 - b_2)(a_4 - b_2)(a_1 - b_3)(a_2 - b_3)(a_3 - b_3)(a_4 - b_3)(a_1 - b_4)(a_2 - b_4)(a_3 - b_4)(a_4 - b_4)$$

$$W_{17}/W_{16} = a_1 + a_2 + a_3 + a_4$$

$$W_{18}/W_{16} = a_1 a_2 + a_1 a_3 + a_2 a_3 + a_1 a_4 + a_2 a_4 + a_3 a_4$$

$$W_{19}/W_{16} = a_1 a_2 a_3 + a_1 a_2 a_4 + a_1 a_3 a_4 + a_2 a_3 a_4$$

$$W_{20}/W_{16} = a_1 a_2 a_3 a_4$$

$$W_{21}/W_{16} = a_1 a_2 a_3 a_4 (a_1 + a_2 + a_3 + a_4)$$

$$W_{22}/W_{16} = a_1 a_2 a_3 a_4 (a_1 a_2 + a_1 a_3 + a_2 a_3 + a_1 a_4 + a_2 a_4 + a_3 a_4)$$

$$W_{23}/W_{16} = a_1 a_2 a_3 a_4 (a_1 a_2 a_3 + a_1 a_2 a_4 + a_1 a_3 a_4 + a_2 a_3 a_4)$$

Identities 3.13. (The generalization of the Girard-Newton Identities, Chen Shuwen, 1997, 2019)

Assume

$$0 \leq n \leq m$$

Let

$$\begin{cases} R_k = a_1^k + a_2^k + \cdots + a_n^k - (b_1^k + b_2^k + \cdots + b_m^k), & \text{for } k \neq 0 \\ R_0 = -\frac{b_1 b_2 \cdots b_m}{a_1 a_2 \cdots a_n} \end{cases}$$

and

$$\begin{cases} T_1 = R_1 \\ T_k = (R_k + T_1 R_{k-1} + T_2 R_{k-2} + \cdots + T_{k-1} R_1)/k, & \text{for } k > 1 \\ T_0 = R_0 \\ T_{-1} = R_0 R_{-1} \\ T_k = (R_0 R_k + T_{-1} R_{k+1} + T_{-2} R_{k+2} + \cdots + T_{k+1} R_{-1})/(-k), & \text{for } k < -1 \end{cases}$$

and

$$\begin{cases} S_k = (-1)^{m-n} T_{k-m+n}, & \text{for } k \leq -1 \\ S_0 = 1 + (-1)^{m-n} T_{n-m} \\ S_k = T_k + (-1)^{m-n} T_{k-m+n}, & \text{for } 1 \leq k \leq m-n \\ S_k = T_k, & \text{for } k \geq m-n+1 \end{cases}$$

and

$$\begin{cases} U_{0,k} = 1 \\ U_{1,k} = S_k \end{cases}$$

and for $2 \leq h \leq n$ and $1 \leq t \leq h-2$

$$\begin{cases} U_{h,hk} = \frac{U_{h-1,(h-1)k}^2 - U_{h-1,(h-1)(k-1)} U_{h-1,(h-1)(k+1)}}{U_{h-2,(h-2)k}} \\ U_{h,hk+t} = \frac{U_{h-1,(h-1)(k+1)} U_{h-1,(h-1)k+t-1} - U_{h-1,(h-1)(k+2)} U_{h-1,(h-1)k-h+t}}{U_{h-2,(h-2)(k+1)}} + \frac{U_{h,hk} U_{h-2,(h-2)(k+1)+t}}{U_{h-2,(h-2)(k+1)}} \\ U_{h,hk+h-1} = \frac{U_{h-1,(h-1)(k+1)} U_{h-1,(h-1)k+h-2} - U_{h-1,(h-1)(k+2)} U_{h-1,(h-1)k-1}}{U_{h-2,(h-2)(k+1)}} \end{cases}$$

and denote

$$W_{nk+t} := U_{n,nk+t} \quad (\text{or } W_t := U_{n,t})$$

then for any k , we have

(1) When $n = 0$

$$S_k = 0$$

(2) When $0 < n \leq m$

$$\begin{cases} W_{nm} = \prod_{i=1}^n \prod_{j=1}^m (a_i - b_j) \\ \frac{W_{nk+1}}{W_{nk}} = a_1 + a_2 + \dots + a_n \\ \frac{W_{nk+2}}{W_{nk}} = a_1 a_2 + a_1 a_3 + \dots + a_{n-1} a_n \\ \frac{W_{nk+3}}{W_{nk}} = a_1 a_2 a_3 + a_1 a_2 a_4 + \dots + a_{n-2} a_{n-1} a_n \\ \dots \dots \\ \frac{W_{nk+n}}{W_{nk}} = a_1 a_2 a_3 \dots a_n \end{cases}$$

Remark. Identities 3.13 and Identities 3.12 are equivalent. Identities 3.12 looks more simple, but need to calculate determinants. Identities 3.13 is much more useful for searching the integer solutions of the Prouhet-Tarry-Escott Problem by using computers. This will be discussed in Chapter 5.

Chapter 4. The Generalization of Ramanujan Identity

To be completed. For some previous result, please see my below site.

<http://eslpower.org/Identity.htm>

Chapter 5. The Generalization of the Prouhet-Tarry-Escott Problem

To be completed. For plenty previous result, please see my below site.

<http://eslpower.org>

<http://eslpower.org/eslp.htm>

<http://eslpower.org/TarryPrb.htm>